A minimal interface problem arising from a two component Bose Einstein condensate via Γ-convergence

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April 25, 2013

Abstract

We consider the energy modeling a two component Bose-Einstein condensate in the limit of strong coupling and strong segregation. We prove the Γ-convergence to a perimeter minimization problem, with a weight given by the density of the condensate. In the case of equal mass for the two components, this leads to symmetry breaking for the ground state. The proof relies on a new formulation of the problem in terms of the total density and spin functions, which turns the energy into the sum of two weighted Cahn-Hilliard energies. Then, we use techniques coming from geometric measure theory to construct upper and lower bounds. In particular, we make use of the slicing technique introduced in [6].

1 Introduction

The aim of this paper is to prove a Γ-convergence result for a functional modeling a two component Bose-Einstein condensate in the case of segregation. We introduce a new formulation of the problem which transforms the two wave functions describing each component of the condensate into total density and spin functions. The new functional in the density and spin variables is given by the sum of two weighted Cahn-Hilliard energies modeling phase transition problems as in the Modica-Mortola problem [26]. In fact, our new functional is strongly related to that of Ambrosio-Tortorelli approaching the Mumford-Shah image segmentation functional [6]. We use techniques coming from geometric measure theory [3, 4, 6, 10] to construct upper and lower bounds for our initial functional and prove Γ-convergence to a perimeter minimization problem, with a weight given by the density of the condensate. There is a large mathematical literature about the segregation patterns for two component Bose Einstein condensates [8, 9, 13, 14, 28, 30]:

regularity of the limiting functions, regularity of the interface, asymptotic behaviour near the interface. All these papers use the limiting equations and do not take into account the trapping potentials and the Γ convergence of the energy as we do.

Before introducing the functional for a two component Bose Einstein condensate, we recall some properties of a single Bose Einstein condensate (BEC). A single BEC is described by the wave function η minimizing the energy

$$E_\varepsilon(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta|^2 + \frac{1}{\varepsilon^2}V(x)|\eta|^2 + \frac{1}{2\varepsilon^2}|\eta|^4 \tag{1.1}$$

where V is the trapping potential, usually taken to be harmonic, that is $V(x) = |x|^2$, ε is a small parameter giving rise to a large coupling constant describing the repulsive self interaction of the condensate. The minimization is performed under the mass constraint $\int_{\mathbb{R}^2} |\eta|^2 = 1$. We define the ground state by

$$E_\varepsilon(\eta_\varepsilon) = \inf_{\int_{\mathbb{R}^2} |\eta|^2=1} E_\varepsilon(\eta)\,, \tag{1.2}$$

which is, up to multiplication by a constant, a real positive function. Let

$$\rho(x) = \max(\lambda^2 - |x|^2, 0) \text{ with } \lambda > 0 \text{ chosen such that } \int_{\mathcal{D}} \rho = 1 \text{ where } \mathcal{D} = B(0,\lambda). \tag{1.3}$$

Then, when ε is small, the ground state η_ε is close to the function $\sqrt{\rho}$ in $\mathcal{D}$, with exponential decay at infinity. Properties of η_ε can be found in [1, 2, 17, 20, 21].

A two component Bose Einstein condensate can be experimentally realized as 2 isotopes of the same atom in different spin states [19] or isotopes of different atoms [27]. They are described by two wave functions u_1 and u_2, respectively representing components 1 and 2. The Gross Pitaevskii energy of the two component condensate is given by

$$\mathcal{E}_\varepsilon(u_1,u_2) = E_\varepsilon(u_1) + E_\varepsilon(u_2) + \frac{1}{2}g_\varepsilon \int_{\mathbb{R}^2} |u_1|^2|u_2|^2\,, \tag{1.4}$$

where E_ε is given by (1.1) and g_ε is the intercomponent coupling strength. The energy is minimized under the mass constraints

$$\int_{\mathbb{R}^2} |u_j|^2 = \alpha_j \quad \text{with} \quad \alpha_j > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 1\,. \tag{1.5}$$

In [24], numerical simulations have been performed to classify the ground states according to the values of ε, g_ε and also the rotational velocity. For ε small and g_ε large, the numerical evidence is that, for $\alpha_1 = \alpha_2 = 1/2$, the preferred ground state is such that each component is asymptotically located in a half disk with a local inverted parabola profile. If $\alpha_1 \neq \alpha_2$, they occupy sections in a disk, the area of which is proportional to α_i. In particular, when neither α_i is too small, this configuration has less energy than a disk vs annulus configuration, which also provides segregation but preserves symmetry. Observation of symmetry breaking has also been obtained experimentally very recently [25]. The breaking of symmetry has been analyzed in [29] in a different limit, namely in the case ε large and g_ε large.

Here, we assume strong coupling between components, that is, $g_\varepsilon \to \infty$, and we study the regime

$$g_\varepsilon \varepsilon^2 \to +\infty \quad \text{and} \quad \varepsilon \to 0\,. \tag{1.6}$$

A trick introduced in [24] is to use a spin formulation also called the nonlinear sigma model. In our special setting, since the ground states are non vanishing real functions, this amounts to defining

$$v := \frac{\sqrt{|u_1|^2+|u_2|^2}}{\eta_\varepsilon} \quad \text{and} \quad \frac{\varphi}{2} := \mathrm{Arg}\left(\frac{|u_1|+i|u_2|}{\sqrt{|u_1|^2+|u_2|^2}}\right), \tag{1.7}$$

where η_ε is defined in (1.2). The definition of φ implies that $|u_1|^2-|u_2|^2 = \eta_\varepsilon^2 v^2\cos\varphi$. The mass constraints (1.5) can be written as

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2 v^2 = \alpha_1+\alpha_2 = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^2} \eta_\varepsilon^2 v^2 \cos\varphi = \alpha_1-\alpha_2\,. \tag{1.8}$$

We point out that $\cos\varphi$ corresponds to the third component of the spin function. Because there is no rotation in the system, the ground states are, up to multiplication by a complex number of modulus one, positive functions. Thus, the second component of the spin is zero and the first one is $\sin\varphi$.
Since the components are expected to segregate, the expected behaviour is thus that v tends to 1 except on a transition line corresponding to the interface between the two components, while φ tends to 0 on component 1 and π on component 2. This is what we want to analyze rigorously.
We split the energy into its main contributions and will prove that

$$\mathcal{E}_\varepsilon(u_1,u_2) = E_\varepsilon(\eta_\varepsilon) + F_\varepsilon(v) + G_\varepsilon(v,\varphi) \tag{1.9}$$

where E_ε is given by (1.1), η_ε is the ground state of E_ε and

$$F_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^2} \eta_\varepsilon^2|\nabla v|^2 + \frac{1}{2\varepsilon^2}\eta_\varepsilon^4\{1-v^2\}^2\,, \tag{1.10}$$

$$G_\varepsilon(v,\varphi) = \frac{1}{8}\int_{\mathbb{R}^2} \eta_\varepsilon^2 v^2\,|\nabla\varphi|^2 + \eta_\varepsilon^4 v^4\,\tilde{g}_\varepsilon\{1-\cos^2(\varphi)\} \tag{1.11}$$

and $\tilde{g}_\varepsilon = g_\varepsilon\left(1-\frac{1}{g_\varepsilon\varepsilon^2}\right)$. Since η_ε^2 converges to ρ given by (1.3) in $\mathcal{D}$, the limits of F_ε and G_ε can be analyzed as the limits of

$$\frac{1}{2}\int_{\mathcal{D}} \rho|\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\rho^2\{1-v_\varepsilon^2\}^2 \tag{1.12}$$

$$\frac{1}{8}\int_{\mathcal{D}} \rho v_\varepsilon^2\,|\nabla\varphi_\varepsilon|^2 + \rho^2 v_\varepsilon^4\,\tilde{g}_\varepsilon\{1-\cos^2(\varphi_\varepsilon)\}. \tag{1.13}$$

These two energies are of Modica Mortola types with a weight which vanishes on the boundary of $\mathcal{D}$. Given the definition of φ_ε, there is a domain where $\cos\varphi_\varepsilon$ tends to 1 (asymptotic region of component 1) and a domain where $\cos\varphi_\varepsilon$ tends to -1 (asymptotic region of component 2), and thus a transition region exists between the two domains. Two options exist for v_ε:

- either v_ε goes to 1 everywhere, which makes the first energy small and the second energy of order $\sqrt{\tilde{g}_\varepsilon}$,

- or v_ε goes to zero on the transition line where $\cos\varphi$ varies from $+1$ to -1: this makes the second energy of lower order and the first energy of order C/ε.

Because of our hypothesis that $\varepsilon^2\tilde{g}_\varepsilon$ tends to infinity, it is the second scenario which costs less energy. Though v_ε goes to 1 on each component, it has a transition region of size ε where it goes sharply to zero. The second energy is of lower order and cannot be seen in the limit. It has just the effect of creating a small region around the interface where v_ε is small. The first energy can be analyzed with techniques coming from [6] and, once the rescaling in ε is made, the Γ-limit comes from the problem on lines:

$$I(x) = \inf\left\{\frac{1}{2}\int_0^\infty \rho(x)(w')^2 + \frac{1}{2}\rho(x)^2(1-w^2)^2\,;\, w \in \mathrm{Lip}(\mathbb{R}_+)\,, w(0)=0 \text{ and } w(+\infty)=1\right\}.$$

Using the Euler-Lagrange equation associated with I, we shall see that for $x \in \mathcal{D}$, the infimum is attained by the function

$$w^x(t) = \tanh\left(\sqrt{\frac{\rho(x)}{2}}\,t\right),$$

and we shall have

$$I(x) = \sigma\rho(x)^{3/2} \text{ with } \sigma = \frac{1}{\sqrt{2}}\int_0^1 \{1-t^2\}\,dt\,. \tag{1.14}$$

This means that w^x is the optimal profile transition at the point x, and that $\sigma\rho(x)^{3/2}$ is the minimum energy needed by w, to go from 0 to 1 at x. In the 1D direction, this provides a weight $2\sigma\rho(x)^{3/2}$ because as $\varepsilon \to 0$, v_ε goes from 1 to 0 on one side of the interface between the two components, and from 0 to 1 on the other side. Therefore, we expect the limit to be defined as the integral on the interface where φ goes from 0 to π of the function $2\sigma\rho(x)^{3/2}$. This requires a precise mathematical definition for this interface. We define X as the space of functions $\varphi \in BV_{\mathrm{loc}}(\mathcal{D}\,;\{0,\pi\})$ such that

$$\int_{\mathbb{R}^2} \rho\cos\varphi = \alpha_1 - \alpha_2\,. \tag{1.15}$$

We will prove the Γ-convergence of $\varepsilon(\mathcal{E}_\varepsilon(\cdot,\cdot) - E_\varepsilon(\eta_\varepsilon))$ to $\mathcal{F}$ given in X by

$$\mathcal{F}(\varphi) = \frac{2\sigma}{\pi}\int_{\mathcal{D}} \rho^{3/2}\,|D\varphi|\,.$$

The limiting energy $\mathcal{F}$ measures the length, with a weight of $\rho^{3/2}$, of the interface between the two phases of φ. Each phase of φ corresponds to one component of the totally segregated two-component limiting condensate. Notice that when $\mathcal{F}(\varphi)$ is finite, $\{\varphi=\pi\}$ has finite perimeter in compact subsets of $\mathcal{D}$, and

$$\mathcal{F}(\varphi) = 2\sigma\int_{\mathcal{D}\cap\partial^*\{\varphi=\pi\}} \rho^{3/2}d\mathcal{H}^1 = 2\sigma\int_{\mathcal{D}\cap S\varphi}\rho^{3/2}d\mathcal{H}^1\,.$$

Here $\partial^*\{\varphi=\pi\}$ stands for the reduced boundary of $\{\varphi=\pi\}$ and $S\varphi$ is the complement of the Lebesgue points of φ, that is,

$$S\varphi = \left\{x \in \mathcal{D}\,;\, \nexists\, t \in \mathbb{R} \text{ such that } \lim_{r\to 0^+}\frac{1}{\pi r^2}\int_{B_r(x)}|\varphi(y)-t|\,dy = 0\right\}.$$

We refer to [5, 15, 18] for the geometric measure theory concepts. We also refer to [3] for an introduction to the theory of Γ-convergence and to the Modica-Mortola theorem by G. Alberti.
We now state our main theorem:

Theorem 1.1. *Let us assume that $V(x) = |x|^2$, and let*

$$\mathcal{H} = \left\{ (u_1, u_2) \in H^1(\mathbb{R}^2; \mathbb{R}) \times H^1(\mathbb{R}^2; \mathbb{R}) \,, \int_{\mathbb{R}^2} V(u_1^2 + u_2^2) < \infty \,, (u_1, u_2) \text{ satisfies (1.5)} \right\} .$$

The functional $\varepsilon(\mathcal{E}_\varepsilon(\cdot,\cdot) - E_\varepsilon(\eta_\varepsilon))$ Γ-converges with respect to the $L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D})$ distance to $\mathcal{F}(\varphi)$, in the following sense:
(Compactness) *for every sequence $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0}$ of minimizers of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$ such that*

$$\sup_{\varepsilon>0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon) \right) < +\infty \,, \tag{1.16}$$

there exists $\varphi \in X$ and a (not relabeled) subsequence such that

$$(u_{1,\varepsilon}, u_{2,\varepsilon}) \to \sqrt{\rho} \left(\mathbf{1}_{\{\varphi=0\}}, \mathbf{1}_{\{\varphi=\pi\}} \right) \qquad \text{in} \qquad L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D}) \,; \tag{1.17}$$

and ***(Lower bound inequality)***

$$\liminf_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon) \right) \geq \mathcal{F}(\varphi) \,. \tag{1.18}$$

(Upper bound inequality) *For every $\varphi \in X$, there exists a sequence $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0} \subset \mathcal{H}$, converging as $\varepsilon \to 0$ to $\sqrt{\rho} \left(\mathbf{1}_{\{\varphi=0\}}, \mathbf{1}_{\{\varphi=\pi\}} \right)$ in $L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D})$, such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon) \right) \leq \mathcal{F}(\varphi) \,. \tag{1.19}$$

We point out that we only prove the Γ-convergence at the level of minimizers of $\mathcal{E}_\varepsilon$. Indeed, minimizers of the functional have the property that they are positive functions which do not vanish. Therefore, this property allows the definition of (v, φ) through (1.7).
As usual, the Γ-convergence theorem implies the convergence of the energy of the ground states:

Corollary 1.2. *If $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0}$ is a sequence of minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$, then*

$$\lim_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon) \right) = \inf_X \mathcal{F} \,. \tag{1.20}$$

A study of the ground states of $\mathcal{F}$ allows us to prove symmetry breaking when neither α_i is too small:

Corollary 1.3. *There exists δ_0 of order 0.15, such that if $\alpha_1 \in [\delta_0, 1-\delta_0]$, then for ε sufficiently small, the minimizers $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$ are not radial.*

Remark 1.4. *Our main theorem remains true when V is any trapping potential for which we have good estimates for the ground state η_ε, namely the estimates in Proposition 2.1.*

1.1 Links with related problems

The segregation behaviour in two component condensates has been widely studied: regularity of the wave function [14, 28, 30], regularity of the interface [13], asymptotic behaviour near the interface [8, 9]. The main difference with these references is that, on the one hand, we use mainly the energy instead of the equation and, on the other hand, we do not switch off the trapping potential by blowing up the problem near the interface or by considering a bounded domain with no trapping. Indeed, we consider the limit where ε goes to zero at the same time as $g_\varepsilon \varepsilon^2$ going to infinity, so that it is the trapping potential which provides the leading order behaviour of the wave function through the inverted parabola profile ρ. In all the previous quoted references, ε is set to 1, so that in the limit g_ε large, the trapping potential is not present, and the limiting profile is 1. We deal with the trapping potential by a proper division of the limiting wave function which allows to express nicely the energy using a trick introduced by [22]. Nevertheless, our proofs which rely on energy considerations also provide information for the case $\rho = 1$.
In [31], the authors fix a point x_∞ on the interface ∂A, and consider a sequence x_ε tending to x_∞ such that $u_{1,\varepsilon}(x_\varepsilon) = u_{2,\varepsilon}(x_\varepsilon) = m_\varepsilon$. An open question in [31] is to prove in 2D that $g_\varepsilon m_\varepsilon^4$ stays bounded. This may be obtained with our technique since in our case m_ε is probably related to the minimum of v_ε. We detail this remark in Section 5.3.

1.2 Main ideas in the proof

Let us now give more details on the proof.
The proof consists of upper and lower bounds, that we construct for the functional $\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) = \varepsilon\,(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\,)$.
For the upper bound, we choose the set A where asymptotically u_2 will be ρ. In a first step, we assume that $\varphi = \pi \mathbf{1}_A$, where A is an open bounded subset of $\mathbb{R}^2$ with smooth boundary such that $\mathcal{H}^1(\partial A \cap \partial \mathcal{D}) = 0$. The test function φ_ε is matched between 0 in a subdomain of $\mathcal{D} \setminus \bar{A}$ to π in a subdomain of A, using a transition region of size $\varepsilon t_\varepsilon$. In order to approximate the optimal 1 dimensional profile that solves $I(y)$, we define

$$w_{\varepsilon,T} = \begin{cases} m_\varepsilon & \text{in} \quad (0, t_\varepsilon) \\ \tanh & \text{in} \quad (t_\varepsilon, T) \\ h & \text{in} \quad (T, T + {}^1\!/\!{}_T) \\ 1 & \text{in} \quad (T + {}^1\!/\!{}_T, +\infty), \end{cases}$$

where $t_\varepsilon = \tanh m_\varepsilon$ and h is a polynomial which matches smoothly tanh to 1. Then we define

$$w^y_{\varepsilon,T}(t) = w_{\varepsilon,T}\left(\sqrt{\frac{\rho(y)}{2}}\, t\right),$$

for $t = d(x)/\varepsilon < CT$, and $d(x)$ is the distance to the boundary. In order to construct v_ε, we need a partition of unity for ∂A, where we match the functions $w^{y_i}_{\varepsilon,T}$, as y_i varies along this partition. For this v_ε, we can estimate F_ε with techniques similar to those of Modica Mortola [26], and to the adaptation of these techniques to problems with weight by Bouchitté [10]. Because ρ vanishes, we cannot use directly the results of Bouchitté and we need precise estimates on the behaviour of η_ε near the boundary. Since $w_{\varepsilon,T}$ is the optimal profile for the 1D version of (1.12), there is a transition from 1 to 0 and a transition from 0 to 1 and we find an upper bound which is $2\int_{\partial A} I(y)\, dy$. Then we prove

that for this test function, $G_\varepsilon(v_\varepsilon, \varphi_\varepsilon)$ is lower order: indeed, the transition layer for φ is is of order $\varepsilon t_\varepsilon$, so much smaller than the one of v_ε. Hence in G_ε, v_ε can be approximated by m_ε. We choose $m_\varepsilon^4 = \varepsilon^2 g_\varepsilon$, which tends to 0, and makes G_ε of lower order.
This provides the upper bound for an open bounded subset A with smooth boundary such that $\mathcal{H}^1(\partial A \cap \partial \mathcal{D}) = 0$. We show in the appendix that for any $\varphi \in X$, $\{\varphi = \pi\}$ can be approximated by sets A which are open bounded subsets of $\mathbb{R}^2$ with smooth boundary such that $\mathcal{H}^1(\partial A \cap \partial \mathcal{D}) = 0$ and that the mass constraints can be satisfied for the approximating $u_{1,\varepsilon}, u_{2,\varepsilon}$.
The difficulty in the lower bound is to prove that v_ε goes to zero on a line and that it provides a positive lower bound. Indeed, the usual Modica-Mortola bound would imply that v_ε goes to 1 almost everywhere and the lower bound is 0. We have to use G_ε and the upper bound to prove that v_ε has a transition to 0 and that $\cos^2 \varphi_\varepsilon$ tends to 1. Hence, because of the mass constraint, we get two regions where asymptotically φ_ε is 0 and π. To analyze the behaviour of v_ε, we use the slicing method introduced in [6] (see also [11]). This consists in looking at the transition for v_ε in one dimensional slices and get the 1D energy estimate. The use of the energy G_ε is only to prove that v_ε goes to zero. We first prove the lower bound for $\varepsilon F_\varepsilon$ in 1D using the coarea formula, and then in 2D using the slicing method. We get that $\varepsilon F_\varepsilon(v_\varepsilon, \varphi_\varepsilon, E)$ converges to a measure $\mu(E)$ supported in S_φ of density $\rho^{3/2}$ with respect to the $\mathcal{H}^1$ measure. The last part of the proof of the lower bound is inspired by ideas in [4].
We end with a variant of the coarea formula that can be found in [23] Lemma 2.2, and in [10] Proposition 2.

Proposition 1.5. *Let Ω be an open bounded subset of $\mathbb{R}^N$, and $\Psi(x, s, p)$ a Borel function of $\Omega \times \mathbb{R} \times \mathbb{R}^N$, which is sublinear in p. Let u be a Lipschitz continuous function on Ω and denote, for every $t > 0$, $S_t = \{x \in \Omega\,;\, u(x) < t\}$. Then, for almost every $t \in \mathbb{R}$, $\mathbf{1}_{S_t}$ belongs to $BV(\Omega)$ and we have*

$$\int_\Omega \Psi(x, u, Du)\, dx = \int_{-\infty}^{\infty} dt \int_\Omega \Psi(x, t, D\mathbf{1}_{S_t})\,. \tag{1.21}$$

The paper is organized as follows: in Section 2, we present the properties of η_ε. Then in Section 3, we prove the decoupling of energy (1.9) and how to go from the (u_1, u_2) formulation to (v, φ). Section 4 is devoted to the upper bound, and Section 5 to the lower bound. Finally, in Section 6, we prove our main theorem.

1.3 To go further

1.3.1 Analysis of the limiting problem

A natural question is to analyze the limiting problem, that is the ground state of $\mathcal{F}$ under the constraint (1.15). If we define A to be the set where $\cos\varphi = 1$. Then $\int_A \rho = \alpha_1$ and $\int_{\mathcal{D}\setminus A} \rho = \alpha_2$ with $\alpha_1 + \alpha_2 = 1$.
If $\rho = 1$, then the problem of minimizing $\mathcal{F}$ amounts to minimizing $|\partial A|$ under the constraints $|A| = \alpha_1$ and $|\mathcal{D} \setminus A| = \alpha_2 = 1 - \alpha_1$. The Euler-Lagrange equation of the minimization problem yields that the curvature is either 0 or constant, hence A is either a disk, an annulus or a disk sector. The equivalent problem with a weight ρ is open.
If we assume that the solution is either two disks sectors or a disk and an annulus, we can compute explicitly the energy $\mathcal{F}$ and find that if $\alpha_1 = \alpha_2$, then the optimal configuration

is two half disks, while if α_1 is much less then α_2, then the ground state is a disk and an annulus (see Section 6.4). Indeed, the energy of two disk sectors is $3\sigma/2$, while the energy of a disk and annulus is $8\sigma(1-\alpha_1)^{3/4}(1-\sqrt{1-\alpha_1})^{1/2}$ if α_1 corresponds to the mass of the inside disk. If α_1 or $\alpha_2 = 1-\alpha_1$ is to small, then the disk and annulus becomes the preferred configuration. In the case $\alpha_1 = \alpha_2 = 1/2$, it follows from our theorem that symmetry breaking occurs since at the limit, the disk plus annulus configuration does not minimize the energy. These two cases are well illustrated in the experimental observations of [25], figure 4.
We insist on the point that a rigorous analysis of the ground states of $\mathcal{F}$ in X is an interesting open question.

1.3.2 Convergence for $u_{1,\varepsilon}$, $u_{2,\varepsilon}$

The convergence that we have for $(u_{1,\varepsilon},\, u_{2,\varepsilon})$ to $\sqrt{\rho}(\mathbf{1}_{\{\varphi=0\}}, \mathbf{1}_{\{\varphi=\pi\}})$ is very weak. Nevertheless, we expect that on compact subsets of $\mathbf{1}_{\{\varphi=\pi\}}$ or $\mathbf{1}_{\{\varphi=0\}}$, the convergence can be improved. For instance, it would be natural to have similar convergence as that of η_ε to $\sqrt{\rho}$ (that is C^1_{loc}) on these domains.

1.3.3 Case $g_\varepsilon \varepsilon^2$ of order 1

An interesting open question is to deal with the case when $g_\varepsilon\varepsilon^2$ tends to a positive finite constant c_0^2. In this case, F_ε and G_ε become of the same order and we expect that $m = \liminf_{\varepsilon\to 0} v_\varepsilon$ is a positive constant (on the interface where φ varies), instead of being 0. We believe that our techniques still provide an upper bound for the problem. We expect the Γ limit to be

$$\left(2\sigma_m + c_0\frac{\pi}{4}m^3\right)\frac{1}{\pi}\int_{\mathcal{D}} \rho^{3/2}\,|D\varphi|\,.$$

where $\sigma_m = \frac{1}{\sqrt{2}}\int_m^1 (1-t^2)\ dt$.

1.3.4 Case of different scattering lengths

In this paper, we consider that the scattering lengths are the same for both components, that is, in (1.4) it is the same energy E_ε for both components. When the two components result experimentally from different atoms, the two scattering lengths are very close but not equal. This leads to an energy E_ε depending on the component, namely

$$E_{\varepsilon,i}(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta|^2 + \frac{1}{\varepsilon^2}|x|^2|\eta|^2 + \frac{g_i}{2\varepsilon^2}|\eta|^4\,,$$

where g_i is related to the scattering lentght of component i. If $g_1 \neq g_2$, then the leading order Thomas Fermi approximation is no longer the same for each component, namely it is

$$g_i\rho_i = \lambda_i^2 - |x|^2 \text{ in } B_i = B(0,\lambda_i).$$

The limiting problem becomes: find a partition of $B_1 \cup B_2$ into three sets A_1, A_2 and N, such that $u_{i,\varepsilon}^2 \to \rho_i \mathbf{1}_{A_i}$, $\int_{A_i}\rho_i = \alpha_i$ and it minimizes

$$\int_{A_1} |x|^2\rho_1 + \frac{g_1}{2}\rho_1^2 + \int_{A_2} |x|^2\rho_2 + \frac{g_2}{2}\rho_2^2. \tag{1.22}$$

This problem is open and is probably related to the problem of finding a partition of the disk into two subdomains which minimize the sum of the first eigenvalues of the Dirichlet laplacian.
Of course, in our case, since we have $B_1 = B_2$, $\rho_1 = \rho_2$ and $N = \emptyset$, (1.22) does not provide any information at leading order. This is why we have to go to the next order which yields the perimeter minimization problem.

2 Estimates for η_ε

Let η_ε be the ground state defined by (1.2). The ground state is a non vanishing radially symmetric function. It is unique up to multiplication by a constant of modulus one, and satisfies the Gross-Pitaevskii equation

$$-\Delta\eta_\varepsilon + \frac{1}{\varepsilon^2}|x|^2\eta_\varepsilon + \frac{1}{\varepsilon^2}|\eta_\varepsilon|^2\eta_\varepsilon = \frac{\lambda_\varepsilon}{\varepsilon^2}\,\eta_\varepsilon\,. \tag{2.1}$$

The term $\varepsilon^{-2}\,\lambda_\varepsilon$ is the Lagrange multiplier associated with the mass constraint, and the pair $(\eta_\varepsilon, \lambda_\varepsilon)$ is unique among positive solutions of (2.1). As ε tends to 0, η_ε tends to $\sqrt{\rho}$ given by (1.3). Throughout the paper, we will need precise estimates for this convergence. The following proposition, based on previous results in [2, 16, 17, 20, 21], sums up the properties of η_ε. We point out that it follows from [16, 17, 21] that an approximation of η_ε by $\sqrt{\rho}$ holds as close to the boundary of $\mathcal{D}$ as needed and is given by (2.5). We also include an estimate of ρ in terms of the distance to the bulk that will be used in the proofs.

Proposition 2.1. *There are constants $c, C > 0$, $\alpha \in (1/2, 3/5)$ and $\gamma \in (1/2, 3/4)$, such that for ε sufficiently small, ρ, λ being given by (1.3),*

$$\begin{aligned}
E_\varepsilon(\eta_\varepsilon) &\le C/\varepsilon^2\,, && && (2.2)\\
|\lambda_\varepsilon - \lambda| &\le C\,\varepsilon\,|\ln\varepsilon|^{1/2}\,, && && (2.3)\\
\|\eta_\varepsilon - \sqrt{\rho}\|_{C^1(K)} &\le C_K\,\varepsilon^2\,|\ln\varepsilon| && \textit{for } K \subset\subset \mathcal{D}\,, && (2.4)\\
|\eta_\varepsilon(x) - \sqrt{\rho}(x)| &\le C\,\varepsilon^\gamma && \textit{for } x \in B(0\,, \lambda - c\,\varepsilon^\alpha)\,, && (2.5)\\
\eta_\varepsilon(x) &\le C\,\varepsilon^{1/6}\,e^{c\,\varepsilon^{-1/3}(\lambda - |x|)} && \textit{for } x \in \mathbb{R}^2\backslash\mathcal{D}\,, && (2.6)\\
\partial_r\eta_\varepsilon(|x|) &\le 0 && \textit{for } x \in \mathbb{R}^2, && (2.7)\\
\frac{\rho(x)}{\lambda\,dist(x, \partial\mathcal{D})} &\in [1, 2) && \textit{for } x \in \mathcal{D}\,. && (2.8)
\end{aligned}$$

Proof: for the proof of (2.2), one can rewrite the energy as

$$E_\varepsilon(\eta) = E^1_\varepsilon(\eta) + \frac{1}{2\varepsilon^2}\left(\lambda^2 - \frac{1}{2}\int_{\mathcal{D}}\rho^2\right) \tag{2.9}$$

where

$$E^1_\varepsilon(\eta) = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla\eta|^2 + \frac{1}{2\varepsilon^2}\left(|\eta|^2 - \rho(x)\right)^2 + \frac{1}{\varepsilon^2}(\lambda^2 - |x|^2)_-\,|\eta|^2\,,$$

and $(\lambda^2 - |x|^2)_-$ is the negative part of $(\lambda^2 - |x|^2)$. In Theorem 2.1 of [2], it is proved that $E^1_\varepsilon(\eta) \le C|\ln\varepsilon|$. Then (2.2) follows from (2.9) and the fact that $\int_{\mathcal{D}}\rho^2 = 2\lambda^2/3$.

Estimate (2.4) is proved in Proposition 2.2 of [20]. Estimates (2.3) and (2.6) are proved in Theorem 2.1 of [2]. Estimate (2.7) is also proved in Theorem 2.1 of [2], but only in a neighborhood of $\partial\mathcal{D}$. But the proof, however, works in the case $V(x) = |x|^2$ and the estimate holds in all $\mathbb{R}^2$.

We now prove (2.5). For $\lambda > 0$, we define $\tilde{\eta}_{\varepsilon,\lambda}$ as the unique radially symmetric, positive solution of the equation

$$-\varepsilon^2\Delta\eta + (\lambda^2 - |x|^2)\eta + \eta^3 = 0\,. \tag{2.10}$$

The function $\tilde{\eta}_{\varepsilon,\lambda}$ corresponds to a ground state of a BEC without mass constraint. In [16, 17, 20], the behavior of $\tilde{\eta}_{\varepsilon,\lambda}$ is studied. Using the results in Proposition 1.2, Remark 1.3 and Proposition 1.4 in [16], we obtain

$$\tilde{\eta}_{\varepsilon,1}(x) = \varepsilon^{1/3}\nu_0\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right) + \mathcal{O}(\varepsilon)\,,$$

where

$$\nu_0(y) = y^{1/2} - \frac{1}{2}y^{-5/2} + \mathcal{O}_{y\to+\infty}(y^{-11/2})\,, \quad y \in (-\infty, \varepsilon^{-2/3}]\,.$$

Hence, for $x \in B(0,1)$ we obtain

$$|\,\tilde{\eta}_{\varepsilon,1}(x) - \sqrt{1-|x|^2}\,| \le C\,\left(\varepsilon^2(1-|x|^2)^{-5/2} + \varepsilon^4(1-|x|^2)^{-11/2} + \varepsilon\right)\,.$$

In particular, if $x \in B(0\,,\lambda - \varepsilon^\alpha)$ with $\alpha \in (1/2, 3/5)$, we get

$$|\,\tilde{\eta}_{\varepsilon,1}(x) - \sqrt{1-|x|^2}\,| \le C\,\left(\varepsilon^2\,\varepsilon^{-5\alpha/2} + \varepsilon^4\,\varepsilon^{-11\alpha/2} + \varepsilon\right) = \mathcal{O}(\varepsilon^\gamma) \tag{2.11}$$

with $\gamma \in (1/2, 3/4)$. We will use (2.11) to prove (2.5). First, a straight computation shows that defining $\varepsilon_\lambda = \lambda^{-2}\varepsilon$, $\tilde{\eta}_{\varepsilon_\lambda,\lambda}$ solves equation (2.10) with $\lambda = 1$. Hence, considering (2.11), a change of variables gives

$$|\,\tilde{\eta}_{\varepsilon_\lambda,\lambda}(x) - \sqrt{\rho}(x)\,| = \mathcal{O}(\varepsilon^\gamma)\,, \tag{2.12}$$

for $x \in B(0\,,\lambda - (\lambda^{-2}\varepsilon)^\alpha)$. In Proposition 2.2 and Theorem 2.2 in [20], it is proved that

$$\|\nabla\eta_\varepsilon\|_{L^\infty(\mathbb{R}^2)} = \mathcal{O}(\varepsilon^{-1})\,; \tag{2.13}$$

and that

$$\eta_{\varepsilon,\lambda}(x) = \ell_{\varepsilon,\lambda}^{1/2}\;\tilde{\eta}_{\tilde{\varepsilon},\lambda}(\ell_{\varepsilon,\lambda}^{-1}\;x)\,, \tag{2.14}$$

where

$$\ell_{\varepsilon,\lambda} = \left(1 + \frac{\varepsilon\lambda_\varepsilon}{\lambda}\right) \qquad \text{and} \qquad \tilde{\varepsilon} = \ell_{\varepsilon,\lambda}^{-1}\varepsilon\,.$$

It follows from (2.3) that

$$\ell_{\varepsilon,\lambda} = 1 + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|^{1/2}) \qquad \text{and} \qquad \tilde{\varepsilon} = \varepsilon + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|^{1/2})\,.$$

Hence, using (2.13) and (2.14), we obtain

$$\eta_{\varepsilon,\lambda}(x) = \tilde{\eta}_{\tilde{\varepsilon},\lambda}(x) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2})\,.$$

Putting this last estimate in (2.12), and using that $\gamma \in (1/2, 3/4)$, we obtain

$$|\,\eta_{\varepsilon_\lambda(x),\lambda} - \sqrt{\rho}(x)\,| = \mathcal{O}(\varepsilon^\gamma)\,,$$

for $x \in B(0\,,\lambda - c\,\varepsilon^\alpha)$ with $c > 0$. We derive (2.5) by changing ε_λ by ε in the previous estimate. Finally, writing

$$\frac{\rho(x)}{\lambda\, dist(x, \partial\mathcal{D})} = \frac{(\lambda + |x|)}{\lambda}$$

we get (2.8) for $|x| < \lambda$.

□

3 Rewriting the energy

In this section, we prove equality (1.9), that is, the reformulation of the Gross-Pitaevskii energy of a two component condensate in (1.4), as the weighted Cahn-Hilliard energy for the pair (v, φ) defined by (1.7), plus the energy of the ground state η_ε of a one component condensate. We start by giving the properties of the minimizers of $\mathcal{E}_\varepsilon$ and the properties of the corresponding pairs $(v_\varepsilon, \varphi_\varepsilon)$ defined by (1.7).

Proposition 3.1. ***(i)*** *Let $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0}$ be a sequence of minimizing pairs of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$ satisfying (1.16). Then, each component is a non vanishing smooth function, and there is $C > 0$ such that*

$$\|u_{1,\varepsilon}\|_{L^\infty(\mathbb{R}^2)}\,,\, \|u_{2,\varepsilon}\|_{L^\infty(\mathbb{R}^2)} < C \tag{3.1}$$

for every $\varepsilon > 0$. Moreover, the pairs $(v_\varepsilon, \varphi_\varepsilon)$ are well defined by (1.7), verify the mass constraints (1.8) and we have

$$(v_\varepsilon, \varphi_\varepsilon) \in Lip_{loc}(\mathbb{R}^2\,;\, (0, +\infty) \times [0, \pi]) \tag{3.2}$$

and

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^\infty(K)} < C_K \qquad \text{for every } K \subset\subset \mathcal{D}\,. \tag{3.3}$$

(ii) *Conversely, let $(v, \varphi) \in Lip(\mathbb{R}^2\,;\,(0, +\infty)\times[0, \pi])$ satisfying (1.8) such that $v\,, \nabla v\,, \nabla\varphi \in L^\infty(\mathbb{R}^2)$. Then, defining*

$$u_1 = \eta_\varepsilon\, v\, \cos(\varphi/2) \quad \text{ and } \quad u_2 = \eta_\varepsilon v\, \sin(\varphi/2)\;, \tag{3.4}$$

we have $(u_1, u_2) \in \mathcal{H}$ and $|u_1|^2 + |u_2|^2 > 0$.

Proof: (i) Let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be a minimizer of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$. Since $\mathcal{E}_\varepsilon(|u_{1,\varepsilon}|, |u_{2,\varepsilon}|) \le \mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon})$, the pair of the absolute values satisfies the system

$$-\Delta u_{1,\varepsilon} + \left(\varepsilon^{-2}V + \varepsilon^{-2}u_{1,\varepsilon}^2 + g_\varepsilon u_{2,\varepsilon}^2\right) u_{1,\varepsilon} = \lambda_{1,\varepsilon}\, u_{1,\varepsilon} \tag{3.5}$$

$$-\Delta u_{2,\varepsilon} + \left(\varepsilon^{-2}V + \varepsilon^{-2}u_{2,\varepsilon}^2 + g_\varepsilon u_{1,\varepsilon}^2\right) u_{2,\varepsilon} = \lambda_{2,\varepsilon}\, u_{2,\varepsilon}\,, \tag{3.6}$$

where $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ are the Lagrange multipliers associated with (1.5). The strong maximum principle yields that $|u_{1,\varepsilon}|$ and $|u_{2,\varepsilon}|$ are positive functions. Using standard elliptic regularity, we deduce further that $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ are non vanishing smooth functions. We use an argument in [20] to prove that $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ are uniformly bounded in $\mathbb{R}^2$. Let us define $w = \varepsilon^{-1}|u_{1,\varepsilon}| - \lambda_\varepsilon^{1/2}$. We have $w \in L^3_{loc}(\mathbb{R}^2)$ and $\Delta w \in L^1_{loc}(\mathbb{R}^2)$. Kato's inequality and equation (3.5) give

$$\Delta(w^+) \geq \mathrm{sgn}^+(w)\, \Delta w \geq \varepsilon^{-3}\, \mathrm{sgn}^+(w)\, \varepsilon w\, (\varepsilon w + \varepsilon\lambda_\varepsilon^{1/2})\, (\varepsilon w + 2\varepsilon\lambda_\varepsilon^{1/2}) \geq (w^+)^3 \,.$$

Hence, $-\Delta(w^+) + (w^+)^3 \leq 0$ weakly in $\mathbb{R}^2$ and Lemma 2 in [12] yield $w^+ \leq 0$. We obtain $|u_{1,\varepsilon}| \leq \varepsilon\lambda_\varepsilon^{1/2}$. Multiplying equation (3.5) by $u_{1,\varepsilon}$ and then integrating we find $\lambda_\varepsilon^{1/2} \leq 2\sqrt{\mathcal{E}_\varepsilon(|u_{1,\varepsilon}|, |u_{2,\varepsilon}|)}$. Since $(u_{1,\varepsilon}, u_{2,\varepsilon})$ verifies (1.16), from estimate (2.2), we derive

$$0 < |u_{1,\varepsilon}| \leq \varepsilon \sqrt{\Big(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\Big) + E_\varepsilon(\eta_\varepsilon)} < C \,.$$

We similarly prove that $0 < |u_{2,\varepsilon}| < C$, so (3.1) is proved. Since $\eta_\varepsilon > 0$, $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ do not vanish in $\mathbb{R}^2$, the pairs $(v_\varepsilon, \varphi_\varepsilon)$ are well defined by (1.7) and $v_\varepsilon > 0$. Since $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ are smooth, v_ε and φ_ε are locally Lipschitz functions so (3.2) holds. The definition of v_ε and (1.5) give

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 = \alpha_1 + \alpha_2 \,.$$

From the definition of φ_ε, we infer that

$$\cos(\varphi_\varepsilon) = \frac{|u_{1,\varepsilon}|^2 - |u_{2,\varepsilon}|^2}{|u_{1,\varepsilon}|^2 + |u_{2,\varepsilon}|^2} \,, \tag{3.7}$$

which, together with (1.5), yields

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon = \alpha_1 - \alpha_2 \,.$$

Hence, $(v_\varepsilon, \varphi_\varepsilon)$ satisfies (1.8). Finally, the estimate (2.4) gives $\eta_\varepsilon \geq c_K > 0$ in $K \subset\subset \mathcal{D}$, so (3.1) yields (3.3).

(ii) Consider (v, φ) as in the statement and define (u_1, u_2) by (3.4). Since (v, φ) verifies (1.8), relation (3.4) gives

$$\int_{\mathbb{R}^2} |u_1|^2 + |u_2|^2 = \int_{\mathbb{R}^2} \eta_\varepsilon^2 v^2 = \alpha_1 + \alpha_2$$

and

$$\int_{\mathbb{R}^2} |u_1|^2 - |u_2|^2 = \int_{\mathbb{R}^2} \eta_\varepsilon^2 v^2 \cos^2\varphi = \alpha_1 - \alpha_2 \,.$$

Thus, (u_1, u_2) verifies (1.5). We have $|u_1|^2 + |u_2|^2 > 0$. Indeed, if it was not the case, since $v > 0$ then φ should take simultaneously the values 0 and π. Since $v \in L^\infty(\mathbb{R}^2)$, bounds (2.5) and (2.6) on η_ε give

$$\int_{\mathbb{R}^2} V\left(|u_1|^2 + |u_2|^2\right) \leq C \int_{\mathbb{R}^2} V\eta_\varepsilon^2 < +\infty \,.$$

We compute

$$|\nabla u_1|^2 \le C\left(v^2|\nabla\eta_\varepsilon|^2 + \eta_\varepsilon^2|\nabla\eta_\varepsilon|^2 + v^2\eta_\varepsilon^2|\nabla\varphi|^2\right).$$

The right hand side of the inequality is integrable in $\mathbb{R}^2$ because $v\,,\nabla v\,,\nabla\varphi \in L^\infty(\mathbb{R}^2)$ and $\eta_\varepsilon \in H^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Thus, $u_{1,\varepsilon} \in H^1(\mathbb{R}^2)$. We prove similarly that $u_{2,\varepsilon} \in H^1(\mathbb{R}^2)$. We have proved that $(u_{1,\varepsilon}, u_{2,\varepsilon}) \in \mathcal{H}$.

□

We now prove the rewriting of the energy.

Proposition 3.2. *Let $(u_1, u_2) \in \mathcal{H}$ satisfying $|u_1|^2 + |u_2|^2 > 0$. Defining (v, φ) by (1.7) we have*

$$\mathcal{E}_\varepsilon(u_1, u_2) = E_\varepsilon(\eta_\varepsilon) + F_\varepsilon(v) + G_\varepsilon(v, \varphi)\,,$$

where E_ε, F_ε and G_ε are given respectively by (1.1), (1.10) and (1.11).

Proof: since $|u_1|^2 + |u_2|^2 > 0$, the pair (v, φ) is well defined. The definitions of v and φ yield

$$|u_1| = \eta_\varepsilon\, v\, \cos(\varphi/2) \qquad \text{and} \qquad |u_2| = \eta_\varepsilon\, v\, \sin(\varphi/2)\,, \tag{3.8}$$

which give

$$\begin{aligned} |u_1|^2 + |u_2|^2 &= \eta_\varepsilon^2 v^2 \\ |u_1|^2|u_2|^2 &= \frac{1}{4}\eta_\varepsilon^4 v^4\{1 - \cos^2\varphi\} \\ |u_1|^4 + |u_2|^4 &= \frac{1}{2}\eta_\varepsilon^4 v^4\{1 + \cos^2\varphi\}\,. \end{aligned} \tag{3.9}$$

Since u_1 and u_2 are real and do not change sign, we have $|\nabla u_1|^2 = |\nabla|u_1||^2$ and $|\nabla u_2|^2 = |\nabla|u_2||^2$. The relations in (3.8) give then

$$|\nabla u_1|^2 + |\nabla u_2|^2 = |\nabla(v\eta_\varepsilon)|^2 + \frac{1}{4}(v\eta_\varepsilon)^2|\nabla\varphi|^2\,. \tag{3.10}$$

Replacing (3.9) and (3.10) in $\mathcal{E}_\varepsilon(u_1, u_2)$ we get

$$\begin{aligned} \mathcal{E}_\varepsilon(u_1, u_2) &= \frac{1}{2}\int |\nabla(v\eta_\varepsilon)|^2 + \frac{1}{\varepsilon^2}V\eta_\varepsilon^2 v^2 \qquad (3.11)\\ &+ \frac{1}{2}\int \frac{1}{4}v^2\eta_\varepsilon^2|\nabla\varphi|^2 + \frac{1}{4\varepsilon^2}\eta_\varepsilon^4 v^4\{1 + \cos^2\varphi\} + \frac{1}{4}g_\varepsilon\, \eta_\varepsilon^4 v^4\{1 - \cos^2\varphi\}\,. \end{aligned}$$

The previous formulation of the energy is the one given by the spin formulation (see the introduction and [24]). We now show how the phase transition model is obtained. Performing an integration by parts, using (2.1) and the first mass constraint in (1.8), we obtain

$$
\begin{aligned}
\int |\nabla(v\eta_\varepsilon)|^2+\frac{1}{\varepsilon^2}V\eta_\varepsilon^2v^2 &= \int v^2\eta_\varepsilon\Big(-\Delta\eta_\varepsilon+\frac{1}{\varepsilon^2}V\eta_\varepsilon\Big)+\eta_\varepsilon^2|\nabla v|^2\\
&= \int v^2\eta_\varepsilon\Big(-\Delta\eta_\varepsilon+\frac{1}{\varepsilon^2}V\eta_\varepsilon+\frac{1}{\varepsilon^2}\eta_\varepsilon^3\Big)-\frac{1}{\varepsilon^2}\eta_\varepsilon^4v^2+\eta_\varepsilon^2|\nabla v|^2\\
&= \frac{\lambda_\varepsilon}{\varepsilon^2}\int v^2\eta_\varepsilon^2+\int \eta_\varepsilon^2|\nabla v|^2-\frac{1}{\varepsilon^2}\eta_\varepsilon^4v^2 \qquad (3.12)\\
&= \frac{\lambda_\varepsilon}{\varepsilon^2}+\int \eta_\varepsilon^2|\nabla v|^2-\frac{1}{\varepsilon^2}\eta_\varepsilon^4v^2\,.
\end{aligned}
$$

Using again (2.1), together with the mass constraint for η_ε, we have that

$$
\frac{\lambda_\varepsilon}{\varepsilon^2}=2\Big(E_\varepsilon(\eta_\varepsilon)+\frac{1}{4\varepsilon^2}\int\eta_\varepsilon^4\Big)\,. \tag{3.13}
$$

Replacing (3.13) in (3.12), and then (3.12) in (3.11) we get

$$
\begin{aligned}
\mathcal{E}_\varepsilon(u_1,u_2) &= E_\varepsilon(\eta_\varepsilon)+\frac{1}{2}\int\eta_\varepsilon^2|\nabla v|^2+\frac{1}{2\varepsilon^2}\eta_\varepsilon^4\{1-2v^2\}\\
&\quad+\frac{1}{2}\int\frac{1}{4}v^2\eta_\varepsilon^2|\nabla\varphi|^2+\frac{1}{4\varepsilon^2}\eta_\varepsilon^4v^4\{1+\cos^2\varphi\}+\frac{1}{4}g_\varepsilon\,\eta_\varepsilon^4v^4\{1-\cos^2\varphi\}\,.
\end{aligned}
$$

Completing the square for $\{1-v^2\}$ we get

$$
\begin{aligned}
\mathcal{E}_\varepsilon(u_1,u_2) &= E_\varepsilon(\eta)+\frac{1}{2}\int\eta_\varepsilon^2|\nabla v|^2+\frac{1}{2\varepsilon^2}\eta_\varepsilon^4\{1-v^2\}^2-\frac{1}{2\varepsilon^2}\eta_\varepsilon^4v^4\\
&\quad+\frac{1}{2}\int\frac{1}{4}v^2\eta_\varepsilon^2|\nabla\varphi|^2+\frac{1}{4\varepsilon^2}\eta_\varepsilon^4v^4\{1+\cos^2\varphi\}+\frac{1}{4}g_\varepsilon\,\eta_\varepsilon^4v^4\{1-\cos^2\varphi\}\\
&= E_\varepsilon(\eta)+\frac{1}{2}\int\eta_\varepsilon^2|\nabla v|^2+\frac{1}{2\varepsilon^2}\eta_\varepsilon^4\{1-v^2\}^2\\
&\quad+\frac{1}{2}\int\frac{1}{4}v^2\eta_\varepsilon^2|\nabla\varphi|^2+\frac{1}{4}\eta_\varepsilon^4v^4g_\varepsilon\Big(1-\frac{1}{g_\varepsilon\varepsilon^2}\Big)\{1-\cos^2\varphi\}\,,
\end{aligned}
$$

which finishes the proof.

□

4 Upper bound inequality

In this section, we consider the formulation of the problem in (v,φ) and call

$$
\mathcal{F}_\varepsilon(v,\varphi)=F_\varepsilon(v)+G_\varepsilon(v,\varphi).
$$

We prove here the upper bound inequality for $\varepsilon\mathcal{F}_\varepsilon$:

Proposition 4.1. ***(Upper bound inequality for*** $\varepsilon\mathcal{F}_\varepsilon$***)*** *Let* $\varphi=\pi\mathbf{1}_A\in X$. *There is a sequence of pairs* $(v_\varepsilon,\varphi_\varepsilon)\in Lip(\mathbb{R}^2;(0,1]\times[0,\pi])$, *converging as* $\varepsilon\to 0$ *to* $(1,\varphi)$ *in* $L^1_{loc}(\mathcal{D})\times L^1_{loc}(\mathcal{D})$, *such that*

$$
\limsup_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon)\le\mathcal{F}(\varphi)\,.
$$

The proof is based on Bouchité's paper [10], where he proves the Γ-convergence of an anisotropic phase transition Cahn-Hilliard energy. We point out that our weight η_ε depends on ε and vanishes asymptotically on the boundary of $\mathcal{D}$.
In a first step, we assume that $\varphi = \pi \mathbf{1}_A$, where A is an open bounded subset of $\mathbb{R}^2$ with smooth boundary such that $\mathcal{H}^1(\partial A \cap \partial \mathcal{D}) = 0$. Then, for any $\varphi \in X$ we approximate $\{\varphi = \pi\}$ by this kind of sets. We conclude then thanks to a density argument. We remark that we do not consider here the mass constraints in (1.8).

Before proving the upper bound, we recall some results about sets with smooth boundary, that can be found in Lemmas 3 and 4 of [26]. For an open set $A \subset \mathbb{R}^2$ with smooth, non empty compact boundary, let d be the signed distance to ∂A, defined by

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial A) & \text{if} \quad x \in A \\ -\operatorname{dist}(x, \partial A) & \text{if} \quad x \in \mathbb{R}^2 \backslash A \,. \end{cases}$$

For small $t > 0$, consider the neighborhood of ∂A given by

$$N_t = \{x \in \mathbb{R}^2 \, ; \, |d(x)| < t\} \,,$$

with boundary

$$S_t = \{x \in \mathbb{R}^2 \, ; \, |d(x)| = t\} \,.$$

For $t > 0$ small enough, there is a diffeomorphism Φ between N_t and $\partial A \times]0, t[$ such that

$$\exists b > 0 \,, \quad \det |D\Phi| \geq b \,. \tag{4.1}$$

We denote by $\hat{\Phi}$ the component of Φ in ∂A. Moreover, d is a Lipschitz continuous function in N_t and we have that

$$|Dd| = 1 \qquad \text{a.e. in } N_t \,. \tag{4.2}$$

For small $t > 0$, define the measure

$$\mu_t = \mathcal{H}^1 \llcorner (\mathcal{D} \cap S_t) \,.$$

Notice that $\mu_0 = \mathcal{H}^1 \llcorner (\mathcal{D} \cap \partial A)$. As in Lemma 4 in [26], $\mathcal{H}^1(\partial A \cap \partial \mathcal{D}) = 0$ yields

$$\liminf_{t \to 0} \mu_t(\Omega) \geq \mu_0(\Omega) \,,$$

for every open $\Omega \subset \mathbb{R}^2$, and

$$\lim_{t \to 0} \mu_t(\mathcal{D}) = \mu_0(\mathcal{D}) \,. \tag{4.3}$$

Hence, as $t \to 0$, μ_t converges weakly* to μ_0, which implies

$$\limsup_{t \to 0} \int_{\mathcal{D}} u \, d\mu_t \leq \int_{\mathcal{D}} u \, d\mu_0 \tag{4.4}$$

for every upper semicontinuous function $u : \mathcal{D} \to \mathbb{R}$ with compact support (see Propositions 1.62 and 1.80 in [5]).

Denote $\eta_0 = \sqrt{\rho}$ and for $\varepsilon \geq 0$ define $f_\varepsilon : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}_+$ by

$$f_\varepsilon(x,t,p) = \frac{1}{2}\eta_\varepsilon^2(x)|p|^2 + \frac{1}{4}\eta_\varepsilon^4(x)\{1-t^2\}^2 \,.$$

For $|p| = 1$ and $s \in \mathbb{R}$ we also write $f_\varepsilon(x,t,s) = f_\varepsilon(x,t,sp)$.

The last step in the proof of Proposition 4.1 uses the following Lemma, which proof is given in the appendix.

Lemma 4.2. *Let A be a subset of $\mathcal{D}$ with $\mathbf{1}_A \in BV_{loc}(\mathcal{D})$. There exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ of open bounded subsets of $\mathbb{R}^2$ with smooth boundaries such that:*

(i) $\lim_{k\to\infty} \mathcal{L}^2((A_k \cap \mathcal{D})\Delta A) = 0$,

(ii) $\limsup_{k\to\infty} \int_\mathcal{D} \rho^{3/2}\,|D\mathbf{1}_{A_k}| \le \int_\mathcal{D} \rho^{3/2}\,|D\mathbf{1}_A|$,

(ii) $\int_{A_k\cap\mathcal{D}} \rho = \int_A \rho$ *and $\mathcal{H}^1(\partial\mathcal{D} \cap \partial A_k) = 0$ for k large enough.*

Proof of Proposition 4.1: we first assume that A is an open subset of $\mathbb{R}^2$ with smooth, non empty compact boundary such that

$$\mathcal{H}^1(\partial A \cap \partial\mathcal{D}) = 0\,. \tag{4.5}$$

(Step 1: construction of the pairs of test functions.) For $T > 1$, consider the approximation of the optimal profile

$$w_T = \begin{cases} \tanh & \text{in} \quad (0,T) \\ h & \text{in} \quad (T, T + {}^1\!/\!{}_T) \\ 1 & \text{in} \quad (T + {}^1\!/\!{}_T, +\infty) \end{cases} \,,$$

where h is the unique cubic polynomial such that $h(T) = \tanh(T)$, $h'(T) = \tanh'(T)$, $h(T + {}^1\!/\!{}_T) = 1$ and $h'(T + {}^1\!/\!{}_T) = 0$. Computing explicitly the coefficients of h, we find that w_T is a nondecreasing function in $\mathbb{R}_+$, with uniform C^1-bounds with respect to $T \in (1,\infty)$. We extend w_T to the whole real line by setting $w_T(t) = w_T(-t)$ in $\mathbb{R}_-$. For $\varepsilon \ge 0$, consider $0 < m_\varepsilon \ll \varepsilon$ (to be chosen later), $t_\varepsilon = \tanh^{-1}(m_\varepsilon)$ and define a modification of w_T near zero by

$$w_{\varepsilon,T} = \begin{cases} m_\varepsilon & \text{in} \quad (0, t_\varepsilon) \\ w_T & \text{in} \quad (t_\varepsilon, \infty)\,. \end{cases}$$

Notice that $w_{\varepsilon,T}$ has uniform Lipchitz bounds with respect to $T \in (1,\infty)$ and $\varepsilon \in [0,1)$. We recall that $\mathcal{D} = B(0,\lambda)$ and we denote $\mathcal{D}_\delta = \overline{B(0,\lambda-\delta)}$. For y in $\partial A \backslash \mathcal{D}_\delta$, we define

$$w_{\varepsilon,T}^y(t) = w_{\varepsilon,T}\left(\sqrt{\frac{\rho(y)}{2}}\,t\right)\,,$$

and we write $w_T^y = w_{0,T}^y$. For small $\delta > 0$, we define $R = R_\delta$ by

$$R = (T + {}^1\!/\!{}_T)\sqrt{\frac{2}{\delta\lambda}}\,. \tag{4.6}$$

Since w_T has uniform C^1-bounds with respect to $T \in (1,\infty)$, while ρ is a smooth function in $\mathcal{D}_\delta$, for every $y \in \partial A \cap \mathcal{D}_\delta$, there is an open neighborhood Σ of y in $\partial A \cap \mathcal{D}_\delta$ such that

$$\int_0^R f_0(x, w_T^y(t), (w_T^y)'(t))\, dt \leq \int_0^R f_0(x, w_T^x(t), (w_T^x)'(t))\, dt \, + \, \delta \qquad \forall x \in \Sigma\,, \quad \forall T \geq 1\,.$$

Hence, thanks to the compactness of $\partial A \cap \mathcal{D}_\delta$, there is a finite family $\{\Sigma_i\}_{i=1}^N$ of open disjoint subsets of $\partial A \cap \mathcal{D}_\delta$, and a corresponding family of points $y_i \in \Sigma_i$, such that

$$\mathcal{H}^1 \left(\partial A \cap \mathcal{D}_\delta \setminus \bigcup_{i=1}^N \Sigma_i \right) = 0 \tag{4.7}$$

and

$$\int_0^R f_0(x, w_T^{y_i}(t), (w_T^{y_i})'(t))\, dt \leq \int_0^R f_0(x, w_T^x(t), (w_T^x)'(t))\, dt \, + \, \delta\,, \tag{4.8}$$

for every $x \in \Sigma_i$, $T \geq 1$ and $1 \leq i \leq N$. We will use the functions $w_{\varepsilon,T}^{y_i}$ to define the first test function, so we have to interpolate between the different Σ_i's. Define first $\Sigma_0 = \partial A \backslash \mathcal{D}_\delta$ and $y_0 = (\lambda - \delta, 0) \in \partial \mathcal{D}_\delta$. For small $\ell > 0$ define $\Sigma_i^\ell = \{x \in \Sigma_i\,;\, \mathrm{dist}(x, \partial \Sigma_i) \geq \ell\}$. Clearly,

$$\mathcal{H}^1(\Sigma_i \backslash \Sigma_i^\ell) \to 0 \quad \text{ as } \quad \ell \to 0.$$

In particular, we can take $\ell = \ell_\delta$ such that

$$R\, \mathcal{H}^1(\Sigma_i \backslash \Sigma_i^\ell) = o_{\delta \to 0}(1) \tag{4.9}$$

for every $0 \leq i \leq N$. Consider then $\{\hat{\theta}_i\}_{i=0}^N$ such that $\hat{\theta}_i \in C^\infty(\partial A, [0,1])$,

$$\sum_{i=0}^N \hat{\theta}_i = 1 \quad \text{on} \quad \partial A \qquad \text{and} \qquad \hat{\theta}_i = 1 \quad \text{in} \quad \Sigma_i^\ell\,. \tag{4.10}$$

We deduce a smooth partition of the unity on N_t by setting $\theta_i = \hat{\theta}_i \circ \hat{\Phi}$ and we define

$$v_\varepsilon = \begin{cases} 1 & \text{in} \quad \mathbb{R}^2 \backslash N_{\varepsilon R} \\ \sum_{i=0}^N \theta_i(x)\, w_{\varepsilon,T}^{y_i}\left(\frac{|d(x)|}{\varepsilon}\right) & \text{in} \quad N_{\varepsilon R} \end{cases}.$$

Since $w_{\varepsilon,T}^{y_i}$ is a nondecreasing function, while ρ is a radial decreasing function, (2.8) and the fact that $\mathrm{dist}(y_i, \mathcal{D}) \geq \delta$ yield

$$w_{\varepsilon,T}^{y_i}\big|_{\partial N_{\varepsilon R}} = w_{\varepsilon,T}^{y_i}(R) = w_{\varepsilon,T}\left((T + 1/T) \sqrt{\frac{\rho(y_i)}{\delta\lambda}} \right) \geq w_{\varepsilon,T}(T + 1/T) = 1\,, \tag{4.11}$$

so v_ε is a continuous function. Moreover, since $w_{\varepsilon,T}$ has uniform Lipschitz bounds with respect to $T \in (1, \infty)$ and $\varepsilon \in (0,1)$, there is $C > 0$ such that for ε small enough,

$$\|v_\varepsilon\|_{C^{0,1}(\mathbb{R}^2)} \leq \frac{C}{\varepsilon}\,. \tag{4.12}$$

We also define

$$\varphi_\varepsilon(x) = \begin{cases} \pi & \text{if} \quad x \in A \setminus N_{\varepsilon\tilde{t}_\varepsilon} \\ \xi\left({}^{d(x)}/_{\varepsilon t_\varepsilon}\right) & \text{if} \quad x \in N_{\varepsilon\tilde{t}_\varepsilon} \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus (A \cup N_{\varepsilon\tilde{t}_\varepsilon}) \end{cases} , \tag{4.13}$$

where $\xi(t) = {}^{\pi}/_{2}(1+t)$ and $\tilde{t}_\varepsilon = ({}^{2}/_{\lambda})^{1/2}\, t_\varepsilon$. We clearly have that $(v_\varepsilon, \varphi_\varepsilon) \in Lip(\mathbb{R}^2; (0,1] \times [0,\pi])$, and that $(v_\varepsilon, \varphi_\varepsilon)$ converges as $\varepsilon \to 0$ to $(1, \varphi)$ in $L^1_{\text{loc}}(\mathcal{D}) \times L^1_{\text{loc}}(\mathcal{D})$.

(Step 2: estimating the energy $\varepsilon G_\varepsilon$.) The function φ_ε is constant in $\mathbb{R}^2 \setminus N_{\varepsilon\tilde{t}_\varepsilon}$, so $G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) = G_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,;\, N_{\varepsilon\tilde{t}_\varepsilon})$. Since $w_{\varepsilon,T}$ is a nondecreasing function while ρ has a global maximum at zero, for every $x \in N_{\varepsilon\tilde{t}_\varepsilon}$ we have

$$w^{y_i}_{\varepsilon,T}\left(\frac{|d(x)|}{\varepsilon}\right) \le w^0_{\varepsilon,T}\left(\frac{|d(x)|}{\varepsilon}\right) = w_{\varepsilon,T}\left(\frac{\lambda}{\sqrt{2}}\frac{|d(x)|}{\varepsilon}\right) \le w_{\varepsilon,T}(t_\varepsilon) = m_\varepsilon\,,$$

so $v_\varepsilon \le m_\varepsilon$ in $N_{\varepsilon\tilde{t}_\varepsilon}$. Hence,

$$G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \le \frac{1}{8}\int_{N_{\varepsilon\tilde{t}_\varepsilon}} \eta_\varepsilon^2 m_\varepsilon^2\, |\nabla\varphi_\varepsilon|^2 + \eta_\varepsilon^4 m_\varepsilon^4\, \tilde{g}_\varepsilon\{1 - \cos^2(\varphi_\varepsilon)\}\,.$$

Then, the definitions of φ_ε and $\tilde{g}_\varepsilon$, together with the fact that η_ε is uniformly bounded, yield

$$G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \le C\left(m_\varepsilon^2\,(\varepsilon\tilde{t}_\varepsilon)^{-2} + g_\varepsilon\, m_\varepsilon^4\right)\,\mathcal{L}^2(N_{\varepsilon\tilde{t}_\varepsilon})\,.$$

Using (4.1), we have

$$\begin{aligned} (\varepsilon\tilde{t}_\varepsilon)^{-1}\mathcal{L}^2(N_{\varepsilon\tilde{t}_\varepsilon}) &\le C\,(\varepsilon\tilde{t}_\varepsilon)^{-1}\int_{-\varepsilon\tilde{t}_\varepsilon}^{\varepsilon\tilde{t}_\varepsilon} dt \int_{\partial A} \left|\det{(D\Phi)^{-1}}\right|\, d\mathcal{H}^1 \\ &\le b^{-1}\,\mathcal{H}^1(\partial A) \\ &= \mathcal{O}_{\varepsilon\to 0}(1)\,. \end{aligned}$$

For ε small, we have $\tilde{t}_\varepsilon = ({}^{2}/_{\lambda})^{1/2}\tanh^{-1}(m_\varepsilon) = \mathcal{O}_{\varepsilon\to 0}(m_\varepsilon)$. Hence,

$$G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \le C\left(m_\varepsilon\varepsilon^{-1} + m_\varepsilon^5\, g_\varepsilon\, \varepsilon\right)\,.$$

Taking $m_\varepsilon = (g_\varepsilon\varepsilon^2)^{-1/4}$, $G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \le C g_\varepsilon^{-1/4}\varepsilon^{-3/2}$, and after (1.6) we obtain

$$\varepsilon G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \le C(g_\varepsilon\varepsilon^2)^{-1/4} = o_{\varepsilon\to 0}(1)\,. \tag{4.14}$$

(Step 3: computing the energy $\varepsilon F_\varepsilon$.) Since v_ε is constant out of $N_{\varepsilon R}$, we have

$$\varepsilon F_\varepsilon(v_\varepsilon) = \int_{N_{\varepsilon R}\cap\mathcal{D}_\delta} \phi_\varepsilon + \int_{N_{\varepsilon R}\cap(\mathcal{D}\setminus\mathcal{D}_\delta)} \phi_\varepsilon + \int_{N_{\varepsilon R}\setminus\mathcal{D}} \phi_\varepsilon\,. \tag{4.15}$$

where

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon} f_\varepsilon(x, v_\varepsilon, \varepsilon D v_\varepsilon)\,.$$

Considering (4.12), for ε small enough there is $C > 0$ such that $|\nabla v_\varepsilon| \le {}^{C}/_{\varepsilon}$. Estimates (2.4), (2.7) and (2.8) thus yield

$$
\begin{aligned}
\int_{N_{\varepsilon R}\cap(\mathcal{D}\backslash\mathcal{D}_\delta)} \phi_\varepsilon &\leq C\left(\max_{\partial\mathcal{D}_\delta}\{\rho+\rho^2\}+C_\delta\,\varepsilon^2|\ln\varepsilon|\right)\varepsilon^{-1}\,\mathcal{L}^2(N_{\varepsilon R}\cap(\mathcal{D}\backslash\mathcal{D}_\delta))\\
&\leq C\left(\delta+\varepsilon^2|\ln\varepsilon|\right)\varepsilon^{-1}\,\mathcal{L}^2(N_{\varepsilon R})\,.
\end{aligned}
$$

Using (4.1) we have

$$
\begin{aligned}
\varepsilon^{-1}\,\mathcal{L}^2(N_{\varepsilon R}) &\leq C\,\varepsilon^{-1}\int_{-\varepsilon R}^{\varepsilon R} dt\int_{\partial A}\left|\det(D\Phi)^{-1}\right|\,d\mathcal{H}^1\\
&\leq C\,R\,b^{-1}\,\mathcal{H}^1(\partial A)\,,
\end{aligned}
$$

so (4.6) yields

$$
\lim_{\varepsilon\to 0}\int_{N_{\varepsilon R}\cap(\mathcal{D}\backslash\mathcal{D}_\delta)}\phi_\varepsilon \leq C\,\delta\,R\,b^{-1}\,\mathcal{H}^1(\partial A)=o_{\delta\to 0}(1)\,. \tag{4.16}
$$

Similarly, using (2.6) we have

$$
\begin{aligned}
\int_{N_{\varepsilon R}\backslash\mathcal{D}}\phi_\varepsilon &\leq C\sup_{\mathbb{R}^2\backslash\mathcal{D}}\left(\eta_\varepsilon^2+\eta_\varepsilon^4\right)R\,b^{-1}\,\mathcal{H}^1(\partial A)\\
&\leq C\,\varepsilon^{1/3}\,R\,b^{-1}\,\mathcal{H}^1(\partial A)\\
&= o_{\varepsilon\to 0}(1)\,.
\end{aligned}\tag{4.17}
$$

Now, remember the interpolation from (4.10). We have

$$
\int_{N_{\varepsilon R}\cap\mathcal{D}_\delta}\phi_\varepsilon=\sum_{i=1}^{N}\left(\int_{N_{\varepsilon R}\cap B_i}\phi_\varepsilon+\int_{N_{\varepsilon R}\cap C_i}\phi_\varepsilon\right)\,,
$$

where

$$
B_i=\{x\in\mathcal{D}_\delta\,;\,\hat\Phi(x)\in\Sigma_i^\ell\}\qquad\text{and}\qquad C_i=\{x\in\mathcal{D}_\delta\,;\,\hat\Phi(x)\in\Sigma_i\backslash\Sigma_i^\ell\}\,.
$$

As before, since η_ε is uniformly bounded in $\mathbb{R}^2$ with respect to $\varepsilon\in(0,1)$, (4.1) yields

$$
\begin{aligned}
\int_{N_{\varepsilon R}\cap C_i}\phi_\varepsilon &\leq C\,\frac{1}{\varepsilon}\int_{-\varepsilon R}^{\varepsilon R}dt\int_{\Sigma_i\backslash\Sigma_i^\ell}\left|\det(D\Phi)^{-1}\right|\,d\mathcal{H}^1\\
&\leq C\,R\,b^{-1}\,\mathcal{H}^1(\Sigma_i\backslash\Sigma_i^\ell)\,.
\end{aligned}
$$

Hence, after (4.6) and (4.9), we obtain

$$
\int_{N_{\varepsilon R}\cap C_i}\phi_\varepsilon=o_{\delta\to 0}(1) \tag{4.18}
$$

for every $\varepsilon\in(0,1)$. In $N_{\varepsilon R}\cap B_i$ we have $v_\varepsilon(x)=w_{\varepsilon,T}^{y_i}({}^{|d(x)|}/_{\varepsilon})$. Using (4.2) we write

$$
\int_{N_{\varepsilon R}\cap B_i}\phi_\varepsilon=\int_{N_{\varepsilon R}\cap B_i}|D({}^{|d|}/_{\varepsilon})|(x)f_\varepsilon(x,w_{\varepsilon,T}^{y_i}({}^{|d(x)|}/_{\varepsilon}),(w_{\varepsilon,T}^{y_i})'({}^{|d(x)|}/_{\varepsilon}))\,.
$$

The coarea formula from Proposition 1.5 yields

$$\int_{N_{\varepsilon R}\cap B_i} \phi_\varepsilon = \int_{-R}^{R} dt \int_{\mathcal{D}_\delta\cap B_i} f_\varepsilon(x, w^{y_i}_{\varepsilon,T}(t), (w^{y_i}_{\varepsilon,T})'(t))\, d\mu_{\varepsilon t}(x)\,.$$

We thus have,

$$\int_{N_{\varepsilon R}\cap B_i} \phi_\varepsilon \le \int_{-R}^{R} dt \int_{\mathcal{D}_\delta\cap B_i} f_0(x, w^{y_i}_{T}(t), (w^{y_i}_{T})'(t))\, d\mu_{\varepsilon t}(x) + R^i_\varepsilon + \tilde{R}^i_\varepsilon\,. \tag{4.19}$$

The first error here before comes from the modification of w_T near 0. Using (4.3) and the definition of t_ε we compute

$$\begin{aligned} R^i_\varepsilon &\le C\sqrt{\frac{2}{\rho(y_i)}}\, t_\varepsilon \sup_{t\in(0,\varepsilon R)} \|\mu_t\| \\ &\le C\sqrt{\frac{2}{\delta\lambda}}\, t_\varepsilon \sup_{t\in(0,\varepsilon R)} \|\mu_t\| \\ &= o_{\varepsilon\to 0}(1)\,. \end{aligned}$$

The second error appears when replacing f_ε by f_0, so using estimates (2.4) and (2.8), together with $y_i \in \mathcal{D}_\delta$, there is $C_\delta > 0$ such that

$$R^2_\varepsilon \le C_\delta\, \varepsilon^2\, |\ln\varepsilon|\, R\ \sup\nolimits_{t\in(0,\varepsilon R)} \|\mu_t\| = o_{\varepsilon\to 0}(1)\,.$$

Using Fubini's formula, we rewrite (4.19) as

$$\int_{N_{\varepsilon R}\cap B_i} \phi_\varepsilon \le \int_{\mathcal{D}} \left(\mathbf{1}_{\mathcal{D}_\delta\cap B_i}(x) \int_{-R}^{R} f_0(x, w^{y_i}_{T}(t), (w^{y_i}_{T})'(t))\, dt\right) d\mu_{\varepsilon t}(x) + o_{\varepsilon\to 0}(1)\,.$$

The set $\mathcal{D}_\delta \cap B_i$ is close and the inner integral is a continuous function of x. Hence, the function inside the outer integral is upper semicontinuous function of x. Inequality (4.4) thus yields

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{N_{\varepsilon R}\cap B_i} \phi_\varepsilon &\le \int_{\mathcal{D}} \left(\mathbf{1}_{\mathcal{D}_\delta\cap B_i}(x) \int_{-R}^{R} f_0(x, w^{y_i}_{T}(t), (w^{y_i}_{T})'(t))\, dt\right) d\mu_0(x) \\ &= \int_{\mathcal{D}_\delta\cap\Sigma^\ell_i} \left(\int_{-R}^{R} f_0(x, w^{y_i}_{T}(t), (w^{y_i}_{T})'(t))\, dt\right) d\mu_0(x)\,. \end{aligned}$$

Notice that since μ_0 is supported in ∂A, we replaced B_i by $\mathcal{D}_\delta \cap \Sigma^\ell_i$. From (4.8) and since w_T is an even function, we have

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{N_{\varepsilon R}\cap B_i} \phi_\varepsilon &\le 2\int_{\mathcal{D}_\delta\cap\Sigma^\ell_i} \left(\int_0^R f_0(x, w^x_T(t), (w^x_T)'(t))\, dt + \delta\right) d\mu_0(x) \\ &\le 2\int_{\mathcal{D}_\delta\cap\Sigma^\ell_i} \left(\int_0^\infty f_0(x, w^x_T(t), (w^x_T)'(t))\, dt + \delta\right) d\mu_0(x)\,. \end{aligned} \tag{4.20}$$

(Step 4: upper bound) Putting together (4.14)-(4.18) and (4.20) we get

$$\limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \;\le\; 2\sum_{i=1}^{N} \int_{\mathcal{D}_\delta \cap \Sigma_i^\ell} \left(\int_0^\infty f_0(x, w_T^x(t), (w_T^x)'(t))\, dt \,+\, \delta \right) d\mu_0(x) + o_{\delta\to 0}(1)\,.$$

Now, we take a sequence $T = T_\delta$ such that $T_\delta \to \infty$ as $\delta \to 0$ (notice that (4.11) still holds). Then, Fubini's formula and dominated convergence theorem, together with (4.7) and (4.9), yield

$$\lim_{\delta\to 0} \left(\limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \right) \;\le\; \int_{\mathcal{D}} \left(\int_0^\infty f_0(x, w^x(t), (w^x)'(t))\, dt \right) d\mu_0(x)\,.$$

Remembering the definitions of f_0, μ_0 and w^x, (1.14) yields

$$\lim_{\delta\to 0} \left(\limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \right) \;\le\; 2\sigma \int_{\mathcal{D}\cap\partial A} \rho^{3/2}(x) d\mathcal{H}^1(x)\,.$$

We conclude thanks to a diagonal argument (see Corollary 1.16 in [7]): there exists a sequence $\delta_\varepsilon \to_{\varepsilon\to 0} 0$, such that as $\varepsilon \to 0$, $(v_{\varepsilon,\delta_\varepsilon}, \varphi_{\varepsilon,\delta_\varepsilon})$ converges in $L^1_{\mathrm{loc}}(\mathcal{D}) \times L^1_{\mathrm{loc}}(\mathcal{D})$ to $(1, \varphi)$, and

$$\limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_{\varepsilon,\delta_\varepsilon}(v_{\varepsilon,\delta_\varepsilon}, \varphi_{\varepsilon,\delta_\varepsilon}) \;\le\; 2\sigma \int_{\mathcal{D}\cap\partial A} \rho^{3/2}(x) d\mathcal{H}^1(x)\,.$$

(Step 5: approximation of A by Cacciopoli sets) We end the proof using Lemma 4.2, the proof of which is given in the appendix. We remove the condition (4.5) and we only assume that A is a set with locally finite perimeter in $\mathcal{D}$. Consider $\varphi^k = \pi \mathbf{1}_{A_k}$ with $\{A_k\}_{k\in\mathbb{N}}$ the sequence from Lemma 4.2. From **(i)**, $(1, \varphi^k)$ converges to $(1, \varphi)$ in $L^1(\mathcal{D})$ and we have $\int_{A_k} \rho = \alpha_2$ for k large enough. Hence, from steps (1)-(4), there is a sequence $(v_\varepsilon^k, \varphi_\varepsilon^k) \to (1, \varphi^k)$ in $L^1_{\mathrm{loc}}(\mathcal{D}) \times L^1_{\mathrm{loc}}(\mathcal{D})$ such that

$$\lim_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon^k, \varphi_\varepsilon^k) = \mathcal{F}(\varphi^k)\,.$$

Using **(ii)** from Lemma 4.2 we obtain

$$\limsup_{k\to 0} \left(\lim_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon^k, \varphi_\varepsilon^k) \right) \le \mathcal{F}(\varphi)\,.$$

As in step (4), we conclude thanks to a diagonal argument.

□

5 Lower bound inequality and compactness

The proofs in this section are based on geometric measure theory techniques. We make the lower bound on lines and then use the slicing method, which can be found in [6] or [11]. The last part of the proof of the lower bound is inspired by the ideas in [4].

5.1 Lower bound on lines

Consider an open set $A \subset\subset \mathcal{D}$ and let $\nu \in S^1$ be a fixed direction. We call π_ν the hyperplane orthogonal to ν, and A_ν the projection of A on π_ν. We define the one dimensional slices of A, indexed by $x \in A_\nu$, as

$$A_x = \{t \in \mathbb{R}\,;\, x + t\nu \in A\}\,.$$

For every function f in $\mathcal{D}$, we define f_x as the restriction of f to the slice A_x, defined by $f_x(t) = f(x + t\nu)$. For $(v, \varphi) : A_x \to (0, 1] \times (0, \pi)$, we define the energies

$$\begin{aligned}
F_\varepsilon(v\,; A_x) &= \frac{1}{2}\int_{A_x} \eta_{\varepsilon,x}^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2}\eta_{\varepsilon,x}^4\{1 - v^2\}^2\,,\\
G_\varepsilon(v, \varphi\,; A_x) &= \frac{1}{8}\int_{A_x} \eta_{\varepsilon,x}^2 v^2\,|\nabla\varphi|^2 + \eta_{\varepsilon,x}^4 v^4\,\tilde{g}_\varepsilon\{1 - \cos^2(\varphi)\} \qquad \text{and}\\
\mathcal{F}_\varepsilon(v, \varphi\,; A_x) &= F_\varepsilon(v; A_x) + G_\varepsilon(v, \varphi; A_x)\,.
\end{aligned}$$

Similarly, for $\varphi \in BV(A_x\,; \{0, \pi\})$ we define

$$\mathcal{F}(\varphi\,; A_x) = \frac{2\sigma}{\pi}\int_{A_x} \rho_x^{3/2} d|D\varphi|\,.$$

With the previous notations, we have the following result:

Proposition 5.1. *Let $(v_\varepsilon, \varphi_\varepsilon) \in Lip(A_x\,; (0, +\infty) \times [0, \pi])$ such that*

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^\infty(A_x)} < \infty \tag{5.1}$$

and

$$\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A_x) < \infty\,. \tag{5.2}$$

Then, there is $\varphi \in SBV(A_x\,; \{0, \pi\})$ such that

$$(v_\varepsilon, \varphi_\varepsilon) \to (1, \varphi) \qquad \text{in} \qquad L^1(A_x) \times L^1(A_x) \tag{5.3}$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A_x) \geq \mathcal{F}(\varphi\,; A_x)\,. \tag{5.4}$$

Proof: *(Step 1)* Using that $A \subset\subset \mathcal{D}$ and estimate (2.4), there are $c_1, c_2 > 0$ such that

$$\eta_{\varepsilon,x}^2 > \rho_x - c_1\varepsilon^2|\ln\varepsilon| > c_2 \quad \text{in} \quad A_x\,. \tag{5.5}$$

Hence, the definition of $\mathcal{F}_\varepsilon(\cdot\,; A_x)$ and (5.2) give

$$\int_{A_x} |1 - v_\varepsilon| < \frac{4|A_x|}{c_2^2}\,\varepsilon^2\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A_x) = o_{\varepsilon\to 0}(1)\,,$$

so $v_\varepsilon \to 1$ in $L^1(A_x)$. Similarly, after (5.1) $v_\varepsilon < C$ in A_x, so (1.6) yields

$$\int_{A_x} v_\varepsilon^4 |1-\cos^2(\varphi_\varepsilon)| < \frac{8C}{\tilde g_\varepsilon\, c_2^2}\, \mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;A_x) = o_{\varepsilon\to 0}(1)\,.$$

Hence, up to a (not relabeled) subsequence, $\varphi_\varepsilon \to \varphi$ a.e. in A_x, with $\varphi : A_x \to \{0,\pi\}$. This, together with $A_x \subset\subset \mathcal{D}$, gives $\varphi_\varepsilon \to \varphi$ in $L^1(A_x)$. We have proved (5.3).

(Step 2) We now prove the lower bound for the energy. Let $t_0 \in S\varphi$. For $\delta > 0$ define

$$J_\delta = A_x \cap (t_0-\delta, t_0+\delta)\,,$$

and suppose that

$$\inf_{t\in J_\delta}\left\{\inf_{\varepsilon>0} v_\varepsilon(t)\right\} > c_3 > 0\,. \tag{5.6}$$

Then, for every $\varepsilon > 0$ and every $t \in J_\delta$, $v_\varepsilon(t) > c_3$. Hence, using (5.2), (5.5) and the coarea formula (1.21), there is $C > 0$ such that

$$(c_2^{-1}c_3^{-2} + c_2^{-2}c_3^{-4})\frac{8\,C}{\tilde g_\varepsilon \varepsilon^2} \geq \int_{J_\delta} |\nabla\varphi|^2 + \{1-\cos^2(\varphi)\} \geq \int_0^\pi dt\,\{1-\cos^2(t)\}^{1/2}\int_{J_\delta}|D\mathbf{1}_{W_{\varepsilon,t}}|\,, \tag{5.7}$$

where $W_{\varepsilon,t} = \{t\in A_x\,;\,\varphi_\varepsilon(x) < t\}$. Since φ_ε converges to φ a.e. in A_x, we get

$$\mathbf{1}_{W_{\varepsilon,t}} \to \mathbf{1}_{\{\varphi=0\}} \quad \text{in} \quad L^1(A_x)\,,$$

for a.e. $t \in (0,\pi)$. Hence, the lower semicontinuity of the BV norm with respect to the L^1-convergence, together with (1.6), (5.7) and Fatou's lemma, gives

$$\begin{aligned} 0 \;&\geq\; \int_0^\pi dt\,\{1-\cos^2(t)\}^{1/2}\liminf_{\varepsilon\to0}\int_{J_\delta}|D\mathbf{1}_{W_{\varepsilon,t}}| \\ &\geq\; \frac{\pi}{2}\int_{J_\delta}|D\mathbf{1}_{\{\varphi=0\}}|\,. \end{aligned}$$

Thus,

$$0 = \mathcal{H}^0(J_\delta\cap S\varphi) \geq \mathcal{H}^0(\{t_0\}) = 1\,.$$

This contradiction implies that (5.6) can not be satisfied. We derive that for every $\delta > 0$, we may extract a subsequence (not relabeled), such that exists $\{t_\varepsilon\}_{\varepsilon>0} \subset J_\delta$ with

$$t_\varepsilon \to \tilde t_0 \in \overline{J_\delta} \qquad \text{and} \qquad v_\varepsilon(t_\varepsilon) \to 0 \qquad \text{as } \varepsilon\to 0\,. \tag{5.8}$$

(Step 3) For $\varepsilon > 0$, define $I_\varepsilon^\pm = \{t\in J_\delta\,;\,\pm(t_\varepsilon - t) < 0\}$ and $v_\varepsilon^\pm : J_\delta \to (0,1]$ by

$$v_\varepsilon^\pm(t) = \mathbf{1}_{I_\varepsilon^\pm} v_\varepsilon(t_\varepsilon) + \mathbf{1}_{I_\varepsilon^\mp} v_\varepsilon(t)\,.$$

The definition of $\mathcal{F}_\varepsilon$, estimate (5.5) and the fact that $v_\varepsilon^\pm$ is constant in $I_\varepsilon^\pm$ while equal to v_ε in $I_\varepsilon^\mp$ yield

$$\sqrt{2}\,\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;J_\delta) \geq \int_{J_\delta}\rho_x^{3/2}\Big(|(v_\varepsilon^+)'|\,|1-(v_\varepsilon^+)^2| + |(v_\varepsilon^-)'|\,|1-(v_\varepsilon^-)^2|\Big) + o_{\varepsilon\to0}(1)\,.$$

Using the coarea formula (1.21) we obtain

$$\sqrt{2}\,\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;J_\delta)\geq\int_0^1 dt\,(1-t^2)\int_{J_\delta}\rho_x^{3/2}\Big(|D\mathbf{1}_{V^+_{\varepsilon,t}}|+|D\mathbf{1}_{V^-_{\varepsilon,t}}|\Big)+o_{\varepsilon\to 0}(1)\,,$$

where $V^{\pm}_{\varepsilon,t}=\{t\in J_\delta\,;\,v_\varepsilon^{\pm}<t\}$. Since $t_\varepsilon\to\tilde{t}_0$, $v_\varepsilon(t_\varepsilon)\to 0$ and $v_\varepsilon(t)\to 1$ a.e. in J_δ, $\mathbf{1}_{V^{\pm}_{\varepsilon,t}}\to\mathbf{1}_{I^{\mp}}$ in $L^1(J_\delta)$, where $I^{\pm}=\{t\in J_\delta\,;\,\pm(\tilde{t}_0-t)\leq 0\}$. Hence, the lower semicontinuity of the BV norm with respect to the L^1-convergence and Fatou's lemma give

$$\liminf_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;J_\delta)\geq\frac{1}{\sqrt{2}}\int_0^1 dt\,(1-t^2)\int_J\rho_x^{3/2}\Big(|D\mathbf{1}_{I^-}|+|D\mathbf{1}_{I^+}|\Big)=\sigma\,2\rho_x^{3/2}(\tilde{t}_0)\,. \quad (5.9)$$

Moreover, since $\rho_x^{3/2}\geq c_4>0$ in A_x, we have

$$\liminf_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;J_\delta)\geq 2\sigma c_4>0\,. \quad (5.10)$$

(Step 4) Let $\Gamma=\{t_1,\ldots,t_n\}$, $n\in\mathbb{N}$, be any finite subset of $S\varphi$. For $i\in\{0,1,\ldots,n\}$ we define

$$J_\delta^i=A_x\cap(t_i-\delta,t_i+\delta)\,.$$

Consider $\delta'>0$ such that $J_\delta^i\cap J_\delta^j=\emptyset$ for $i\neq j$ and let $\delta\in(0,\delta')$. From (5.10), we have

$$\frac{c_4}{2\sigma}\liminf_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;A_x)\geq n\,.$$

Therefore, using (5.2) we derive that n is bounded, so $S\varphi$ is a finite set and $\varphi\in SBV(A_x)$.

(Step 5) Finally, write $S\varphi=\{t_1,\cdots,t_N\}$, $N\in\mathbb{N}$. Reasoning as before, for δ' small enough and $\delta\in(0,\delta')$, (5.9) gives

$$\liminf_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;A_x)\geq 2\sigma\sum_{i=1}^N\rho_x^{3/2}(\tilde{t}_i)\,,\qquad\text{with}\qquad\tilde{t}_i\in\overline{J_\delta^i}\,.$$

Since ρ_x is a continuous function, taking the limit $\delta\to 0$ in the previous inequality we obtain

$$\liminf_{\varepsilon\to 0}\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon,\varphi_\varepsilon\,;A_x)\geq 2\sigma\sum_{i=1}^N\rho_x^{3/2}(t_i)=2\sigma\int_{A_x}\rho_x^{3/2}\,d|D\varphi|\,.$$

We have proved (5.4).

□

5.2 The slicing method

Using the slicing method, we now prove the compactness and the lower bound inequality for $\varepsilon\mathcal{F}_\varepsilon$.

Proposition 5.2. ***(Lower bound inequality and compactness for*** $\varepsilon\mathcal{F}_\varepsilon$***)*** *Let* $(v_\varepsilon, \varphi_\varepsilon) \in Lip_{loc}(\mathbb{R}^2\,;\,(0,+\infty)\times[0,\pi])$ *such that*

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^\infty(K)} < C_K \qquad \textit{for } K \subset\subset \mathcal{D} \tag{5.11}$$

and

$$\sup_{\varepsilon>0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) < \infty\,. \tag{5.12}$$

Then, there is $\varphi \in X$ *such that*

$$(v_\varepsilon, \varphi_\varepsilon) \to (1, \varphi) \qquad \textit{in} \qquad L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D}) \tag{5.13}$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,) \geq \mathcal{F}(\varphi)\,. \tag{5.14}$$

Proof: arguing as in the proof of (5.3) of Proposition 5.1, there exists $\varphi : \mathcal{D} \to \{0,\pi\}$ such that (5.13) is satisfied. Consider an open set $A \subset\subset \mathcal{D}$, and fix $\nu \in S^1$. For $x \in A_\nu$, we define $(v_{\varepsilon,x}, \varphi_{\varepsilon,x}) : A_x \to (0,1]\times(0,\pi)$ by

$$(v_{\varepsilon,x}, \varphi_{\varepsilon,x})(t) = (v_\varepsilon, \varphi_\varepsilon)(x + t\nu)\,.$$

Since $A \subset\subset \mathcal{D}$ and since v_ε, η_ε are non vanishing continuous functions, for fixed $\varepsilon > 0$ (5.12) yields

$$\begin{aligned} C \geq \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A) &\geq \frac{\varepsilon}{2}\inf_A \eta_\varepsilon^2 \int_A |\nabla v_\varepsilon|^2 + \frac{\varepsilon}{8}\inf_A v_\varepsilon^2\eta_\varepsilon^2 \int_A |\nabla\varphi_\varepsilon|^2 \\ &\geq c_{\varepsilon,A}\left\{\int_A |\nabla v_\varepsilon|^2 + \int_A |\nabla\varphi_\varepsilon|^2\right\}, \end{aligned}$$

so v_ε and φ_ε belong to $W^{1,2}(A)$. Hence (see [15], Section 4.9.2),

$$v'_{\varepsilon,x}(t) = D_\nu v_\varepsilon(x+t\nu) \qquad \text{and} \qquad \varphi'_{\varepsilon,x}(t) = D_\nu\varphi_\varepsilon(x+t\nu)$$

for a.e. $t \in A_x$, for $\mathcal{L}^1$- a.e. $x \in A_\nu$. Using then $|\nabla v_\varepsilon|^2 \geq |D_\nu v_\varepsilon|^2$, we get the slicing inequality

$$\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A) \geq \int_{A_\nu} \varepsilon\mathcal{F}_\varepsilon(v_{\varepsilon,x}, \varphi_{\varepsilon,x}\,; A_x)\,dx\,. \tag{5.15}$$

From (5.12), for $\mathcal{L}^1$-a.e. $x \in A_\nu$, $\varepsilon\mathcal{F}_\varepsilon(v_{\varepsilon,x}, \varphi_{\varepsilon,x}\,; A_x)$ is uniformly bounded with respect to ε. Thus, after Proposition 5.1, for $\mathcal{L}^1$-a.e. $x \in A_\nu$ there is $\varphi_x \in BV(A_x\,;\{0,\pi\})$ such that

$$(v_{\varepsilon,x}, \varphi_{\varepsilon,x}) \to (1, \varphi_x) \qquad \text{in} \qquad L^1(A_x) \times L^1(A_x) \tag{5.16}$$

and

$$\liminf_{\varepsilon>0} \varepsilon\mathcal{F}_\varepsilon(v_{\varepsilon_x}, \varphi_{\varepsilon,x}\,; A_x) > \mathcal{F}(\varphi_x\,; A_x)\,. \tag{5.17}$$

The function φ defined in (5.13) is the $L^1(A)$ limit of φ_ε, so for $\mathcal{L}^1$-a.e. $x \in A_\nu$, φ_x coincide with the restriction of φ to A_x. Therefore, since the vector ν is taken arbitrarily, $\varphi \in BV(A)$ (see Proposition 6.9 in [4]), and since A is any open relatively compact subset of $\mathcal{D}$, we derive that $\varphi \in BV_{loc}(\mathcal{D})$. Using (5.15), (5.17) Fatou's lemma and Fubini's formula, we also obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) &\geq \int_{A_\nu} \mathcal{F}(1, \varphi_x\,; A_x)\,dx \\
&= \frac{2\sigma}{\pi}\int_{A_\nu} dx \int_{A_x} \rho_x^{3/2} d|D\varphi| \\
&= \frac{2\sigma}{\pi}\int_{\mathcal{D}} \rho_x^{3/2} d(\mathcal{L}^1 \llcorner A_\nu \otimes |D\varphi_x| \llcorner A_x)\,. \qquad (5.18)
\end{aligned}$$

Now, for every $\varepsilon > 0$, let μ_ε be the energy distribution in $\mathcal{D}$ associated with the pair $(v_\varepsilon, \varphi_\varepsilon)$, that is, the positive Radon measure which for every Borel set $E \subset \mathbb{R}^2$ is given by

$$\mu_\varepsilon(E) = \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; E \cap \mathcal{D})\,.$$

From (5.12), the total mass $\|\mu_\varepsilon\|$ is uniformly bounded. De La Vallée Poussin compactness criterion (see [5], page 26) gives then that (up to a subsequence) μ_ε converges weakly* to some finite measure μ on $\mathcal{D}$. We claim that

$$\mu \geq 2\sigma\rho^{3/2} \cdot \mathcal{H}^1 \llcorner S\varphi\,.$$

We will prove this using Besicovitch derivation Theorem (see [5], page 54). First, after (5.12) for every $K \subset\subset \mathcal{D}$ there is $R_K \in (0, \lambda)$ such that

$$0 \leq \mu(K) \leq \mu(B(0, R_K)) \leq \liminf_{\varepsilon\to 0} \mu_\varepsilon(B(0, R_K)) < \infty. \qquad (5.19)$$

Hence, μ is a positive Radon measure in $\mathcal{D}$, and for $\mathcal{H}^1$-a.e. $x \in S\varphi$ the limit

$$f(x) = \lim_{r\to 0^+} \frac{\mu(B_r(x))}{\mathcal{H}^1(B_r(x) \cap S\varphi)} \qquad (5.20)$$

exists, and we have

$$\mu \geq f \cdot \mathcal{H}^1 \llcorner S\varphi\,. \qquad (5.21)$$

Let $x_0 \in S\varphi \cap A$. Since $A \subset\subset \mathcal{D}$, $\overline{B_r(x_0)} \subset \mathcal{D}$ for r small enough. We assume[1] moreover that $\mu(\partial B(x_0, r)) = 0$. Proposition 1.62 in [5] and estimate (5.18) yield

$$\begin{aligned}
\mu(B_r(x_0)) &= \lim_{\varepsilon\to 0} \mu_\varepsilon(B_r(x_0)) \\
&\geq \frac{2\sigma}{\pi}\int_{B_r(x_0)} \rho_x^{3/2} d(\mathcal{L}^1 \llcorner A_\nu \otimes |D\varphi_x| \llcorner A_x) \\
&\geq \frac{2\sigma}{\pi} \inf_{B_r(x_0)} \rho_x^{3/2} \int_{B_r(x)} d(\mathcal{L}^1 \llcorner A_\nu \otimes |D\varphi_x| \llcorner A_x)\,. \qquad (5.22)
\end{aligned}$$

[1] In fact this holds for all r except countably many (see [5], page 29).

In Proposition 5.1 we proved that for $\mathcal{L}^1$-a.e. $x \in A_\nu$, $\varphi_x \in SBV(A_x) \cap L^\infty(A)$, and

$$\int_{A_\nu} dx \int_{A_x} \mathcal{H}^0(S_{\varphi_x}) < \infty\,.$$

Hence, after Theorem 2.3 in [6], $\varphi \in SBV(A) \cap L^\infty(A)$ and

$$\int_{B_r(x_0)} d(\mathcal{L}^1 \llcorner A_\nu \otimes |D\varphi_x| \llcorner A_x) = \pi \int_{B_r(x_0)} |\langle \nu_\varphi, \nu\rangle|\, d\mathcal{H}^1\,, \tag{5.23}$$

where ν_φ is the measure theoretic inner normal to the Caccioppoli set $\{\varphi = \pi\}$. Putting (5.23) in (5.22) we obtain

$$\begin{aligned}\mu(B_r(x_0)) &\geq 2\sigma \inf_{B_r(x_0)} \rho_x^{3/2} \int_{B_r(x_0)} |\langle \nu_\varphi, \nu\rangle|\, d\mathcal{H}^1 \\ &\geq 2\sigma \inf_{B_r(x_0)} \rho_x^{3/2} \inf_{B_r(x_0)\cap S\varphi} |\langle \nu_\varphi, \nu\rangle|\, \mathcal{H}^1(B_r(x_0) \cap S\varphi)\,.\end{aligned}$$

Since $S\varphi$ is a rectifiable set in A, for $\mathcal{H}^1$-a.e. $x_0 \in S\varphi$, $\nu_\varphi(x)$ is continuous in $B_r(x_0)$ for r small enough. Thus, taking $\nu = \nu_\varphi(x_0)$ and since ρ is continuous, we get

$$\lim_{r\to 0^+} \frac{\mu(B_r(x_0))}{\mathcal{H}^1(B_r(x_0) \cap S\varphi)} \geq 2\sigma\rho^{3/2}(x_0)$$

for $\mathcal{H}^1$-a.e. $x_0 \in S\varphi$. Hence, (5.20) and (5.21) yield the claim. The definition of μ gives then

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,; A) = \liminf_{\varepsilon\to 0} \mu_\varepsilon(A) \geq \mu(A) \geq 2\sigma \int_{S\varphi\cap A} \rho^{3/2} d\mathcal{H}^1\,.$$

Finally, taking an increasing sequence $\{A_k\}_{k\in\mathbb{N}}$ with $A_k \subset\subset \mathcal{D}$, we get

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \geq 2\sigma \int_{S\varphi\cap\mathcal{D}} \rho^{3/2} d\mathcal{H}^1\,,$$

which gives (5.14).

□

5.3 Remark about a lower bound for v_ε in the transition zone

We end this section with a discussion about the infinimum of v_ε in the transition zone. Let $\{(v_\varepsilon, \varphi_\varepsilon)\}_{\varepsilon>0}$ be a sequence of minimizers of $\mathcal{F}_\varepsilon$, and let $\varphi \in BV_{\mathrm{loc}}(\mathcal{D}\,; \{0,\pi\})$ be the L^1_{loc}-limit of φ_ε given in (1.17). Let $K \subset\subset \mathcal{D}$ be an open smooth set, with non negligible intersection with $S\varphi$, that is,

$$\mathcal{H}^1(K \cap S\varphi) > 0\,.$$

For every $\varepsilon > 0$, we define

$$m_{\varepsilon,K} = \inf_{x\in K} v_\varepsilon(x)\,.$$

We would like to obtain an upper bound for $m_{\varepsilon,K}$, in connection with an open question in [8], namely

$$m_{\varepsilon,K} \le C_K (g_\varepsilon \varepsilon^2)^{-1/4}\,. \tag{5.24}$$

If we assume that we have the upper and lower inequalities for each $\varepsilon > 0$, that is

$$\varepsilon F_\varepsilon(\tilde{v}_\varepsilon) \le \mathcal{F}(\varphi) \tag{5.25}$$

and

$$\varepsilon F_\varepsilon(\tilde{v}_\varepsilon) \ge \mathcal{F}(\varphi)\,, \tag{5.26}$$

we can give estimates on G_ε in order to obtain the upper bound for $m_{\varepsilon,K}$. So assume that we have (5.25) and (5.26). On the one hand, estimates (2.4) and (2.8) give then

$$\begin{aligned}
\varepsilon G_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,;\,K) &\ge \frac{1}{4}\sqrt{g_\varepsilon \varepsilon^2}\, m_{\varepsilon,K}^3 \left(\inf_K \rho^{3/2} - C_K \varepsilon^2 |\ln \varepsilon^2|\right) \int_K |\nabla \varphi_\varepsilon| \sin \varphi_\varepsilon \\
&\ge C_K \sqrt{g_\varepsilon \varepsilon^2}\, m_{\varepsilon,K}^3 \int_K |\nabla \varphi_\varepsilon| \sin \varphi_\varepsilon\,.
\end{aligned}$$

We claim that the integral here below is bounded away from zero. Indeed, if this not the case, we will have

$$\liminf_{\varepsilon\to 0} \int_K |\nabla \varphi_\varepsilon| \sin \varphi_\varepsilon = 0\,.$$

Hence, since $\varphi_\varepsilon \to \varphi$ in $L^1(K)$, the coarea formula together with the lower semi continuity of the BV norm imply the contradiction

$$0 = \liminf_{\varepsilon\to 0} \int_0^\pi \sin t\, dt \int_K |D\mathbf{1}_{\{\varphi_\varepsilon < t\}}| \ge \int_0^\pi \sin t\, dt \int_K |D\mathbf{1}_{\{\varphi = 0\}}| = 2\,\mathcal{H}^1(S\varphi \cap K)\,.$$

We thus derive that there is $C'_K > 0$ such that

$$\varepsilon G_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,;\,K) \ge C'_K \sqrt{g_\varepsilon \varepsilon^2}\, m_{\varepsilon,K}^3\,. \tag{5.27}$$

In the other hand, by inspection of the proof of Proposition 4.1 (see estimate (4.14)), we see that the pair of test function $(\tilde{v}_\varepsilon, \tilde{\varphi}_\varepsilon)$ satisfies

$$\varepsilon G_\varepsilon(\tilde{v}_\varepsilon, \tilde{\varphi}_\varepsilon\,;\,K) \le C(g_\varepsilon \varepsilon^2)^{-1/4}\,. \tag{5.28}$$

Hence, considering (5.25)-(5.28), together with the fact that $(v_\varepsilon, \varphi_\varepsilon)$ minimizes $\mathcal{F}_\varepsilon$, we obtain

$$\begin{aligned}
2\sigma \int_K \rho^{3/2} |D\varphi| + C_K \sqrt{g_\varepsilon \varepsilon^2}\, m_{\varepsilon,K}^3 &\le \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,;\,K) \\
&\le \varepsilon \mathcal{F}_\varepsilon(\tilde{v}_\varepsilon, \tilde{\varphi}_\varepsilon\,;\,K) \\
&\le 2\sigma \int_K \rho^{3/2} |D\varphi| + C(g_\varepsilon \varepsilon^2)^{-1/4}\,.
\end{aligned}$$

Multiplying both sides of the previous inequality by $(g_\varepsilon \varepsilon^2)^{1/4}$ we find the upper bound (5.24) for $m^3_{\varepsilon,K}$.
However, we are not able to prove (5.25) and (5.26) as such because of the error terms. Indeed, the proof of the upper bound of Theorem 1.1 says that there is a sequence $\{(\tilde{v}_\varepsilon, \tilde{\varphi}_\varepsilon)\}_{\varepsilon>0}$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon F_\varepsilon(\tilde{v}_\varepsilon) \leq \mathcal{F}(\varphi) \,. \tag{5.29}$$

In the proof of (5.29), we first approximate the locally Cacciopoli set $A = \{\varphi = \pi\}$ by characteristics functions of open sets A_k with compact smooth boundary. This gives a small error in terms of $k \in \mathbb{N}$ in the upper bound inequality (5.29). Then for each $k \in \mathbb{N}$, we construct a test function for which (5.29) holds, up to a small error term depending on a parameter $\delta > 0$. In these two steps, we use diagonal extraction arguments in order to get rid of the error terms, so it is not possible to compute them explicitly. Similarly, in the proof of the lower bound of Theorem 1.1, we use the compactness of bounded Radon measures, so we cannot estimate the error term in the lower bound inequality

$$\liminf_{\varepsilon\to 0} \varepsilon F_\varepsilon(\tilde{v}_\varepsilon) \geq \mathcal{F}(\varphi) \,. \tag{5.30}$$

6 Proof of the Γ-convergence for $\varepsilon\left(\mathcal{E}_\varepsilon(\cdot) - E_\varepsilon(\eta_\varepsilon)\right)$

6.1 Proof of the compactness and the lower bound inequality in Theorem 1.1:

let $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0}$ be a sequence of minimizers of $\mathcal{E}_\varepsilon$ in $\mathcal{H}$ satisfying (1.16). From Proposition 3.1**(i)**, the pairs $(v_\varepsilon, \varphi_\varepsilon)$ are well defined by (1.7), belong to $Lip_{loc}(\mathbb{R}^2\,;\,(0,+\infty)\times[0,\pi])$ and satisfy (5.11). Proposition 3.2 yields $\varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) < \infty$. Thus, the hypotheses of Proposition 5.2 are fulfilled by $(v_\varepsilon, \varphi_\varepsilon)$ and we have

$$(v_\varepsilon, \varphi_\varepsilon) \to (1, \varphi) \qquad \text{in} \qquad L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D})$$

with $\varphi \in X$, and

$$\liminf_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon\,) \geq \mathcal{F}(\varphi\,) \,.$$

Equality (1.9) yields then

$$\liminf_{\varepsilon\to 0} \varepsilon\left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) \geq \mathcal{F}(\varphi) \,.$$

Finally, using identity (3.4) we get

$$(u_{1,\varepsilon}, u_{2,\varepsilon}) \to \sqrt{\rho}\,(\mathbf{1}_{\{\varphi=0\}}, \mathbf{1}_{\{\varphi=\pi\}}) \quad \text{in} \quad L^1_{loc}(\mathcal{D}) \times L^1_{loc}(\mathcal{D}) \,.$$

□

In order to prove the upper bound we have to work a little more. We first modify the pairs of test functions from Proposition 4.1 to make them satisfy the mass constraints

(1.8). We prove then that this modification do not change the limit of the energy. We finish by verifying that the pairs of modified test functions are the image by (1.7) of a pair in $\mathcal{H}$, and we conclude using Proposition 4.1.

6.2 Proof of the upper bound inequality in Theorem 1.1:

(Step 1 : Modification of the pairs of test functions) With the notations from the proof of Proposition 4.1, we write $N_\varepsilon = N_{\varepsilon R_\delta}$ and we define $(\check{v}_\varepsilon, \varphi_\varepsilon)$ the sequence of pairs of test functions such that

$$\limsup_{\varepsilon\to 0} \varepsilon\mathcal{F}_\varepsilon(\check{v}_\varepsilon, \varphi_\varepsilon) \le \mathcal{F}(\varphi)\,. \tag{6.1}$$

Consider $\kappa \in C^\infty(\mathbb{R}_+\,;\,[0,1])$ with $supp\,\kappa \subset (0,1)$ and $\kappa = 1$ in $(0, {}^1\!/\!_2)$. Since A is a non empty open set, there is $B_0 = B_{r_0}(x_0) \subset\subset A \cap \mathcal{D}$. For $\ell \in [-1,1]$ and $\tau \in ({}^1\!/\!_2, 1)$, define $\kappa_\varepsilon = \kappa_{\varepsilon,\ell,\tau}$ by

$$\kappa_\varepsilon(x) = \varepsilon^\tau \ell\, \kappa({}^{|x-x_0|}\!/\!_{r_0})\,.$$

We define then $\hat{v}_\varepsilon = \check{v}_\varepsilon + \kappa_\varepsilon$ and $v_\varepsilon = c_\varepsilon \hat{v}_\varepsilon$, with $c_\varepsilon = \|\eta_\varepsilon \hat{v}_\varepsilon\|_2^{-2}$. For ε small enough N_ε and B_0 are disjoints. We estimate

$$\begin{aligned} c_\varepsilon^{-1} &= 1 + \int_{N_\varepsilon\cup B_0} \eta_\varepsilon^2(\hat{v}_\varepsilon^2 - 1) \\ &= 1 + 2\int_{B_0} \eta_\varepsilon^2 \kappa_\varepsilon + \int_{N_\varepsilon} \eta_\varepsilon^2(\check{v}_\varepsilon^2 - 1) + \int_{B_0} \eta_\varepsilon^2 \kappa_\varepsilon^2 \\ &= 1 + 2\int_{B_0} \eta_\varepsilon^2 \kappa_\varepsilon + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{2\tau})\,. \end{aligned}$$

Hence, using that $\tau \in ({}^1\!/\!_2, 1)$ we get $c_\varepsilon^2 = 1 - r_\varepsilon$ with

$$r_\varepsilon = 4\int_{B_0} \eta_\varepsilon^2 \kappa_\varepsilon + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon^\tau)\,. \tag{6.2}$$

Notice that for ε small enough, r_ε may be positive or negative depending on the sign of ℓ.

The definition of $w^y_{\varepsilon,T}$ insures that $v_\varepsilon > 0$. The first mass constraint in (1.8) is immediately satisfied by the definition of v_ε. Remember the definition of φ_ε in (4.13). For the second mass constraint we write

$$c_\varepsilon^{-2}\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon = \int_{\mathbb{R}^2} \eta_\varepsilon^2(\mathbf{1}_{\mathbb{R}^2\setminus(A\cup N_\varepsilon)} - \mathbf{1}_{A\setminus(N_\varepsilon\cup B_0)} + \mathbf{1}_{N_\varepsilon\cup B_0}\,\hat{v}_\varepsilon^2\cos\varphi_\varepsilon)\,.$$

Adding and removing $\mathbf{1}_{N_\varepsilon\setminus A}\,\eta_\varepsilon^2$, $\mathbf{1}_{N_\varepsilon\cup A}\,\eta_\varepsilon^2$ and $\mathbf{1}_{B_0}\,\eta_\varepsilon^2$ in the previous integral, we get

$$c_\varepsilon^{-2}\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon = \int_{\mathbb{R}^2} \eta_\varepsilon^2(\mathbf{1}_{\mathbb{R}^2} - 2\mathbf{1}_A) + \int_{B_0} \eta_\varepsilon^2(\check{v}_\varepsilon + \kappa_\varepsilon)^2 - 1 \tag{6.3}$$

$$+\int_{N_\varepsilon} \eta_\varepsilon^2(\check{v}_\varepsilon^2\cos\varphi_\varepsilon - \mathbf{1}_A + \mathbf{1}_{\mathbb{R}^2\setminus A})\,. \tag{6.4}$$

For the third term in (6.3), we have that η_ε, $\check{v}_\varepsilon$ and $\cos\varphi_\varepsilon$ are bounded while $\mathcal{L}^2(N_\varepsilon) = \mathcal{O}(\varepsilon)$. Hence,

$$\int_{N_\varepsilon} \eta_\varepsilon^2(\check{v}_\varepsilon^2 \cos\varphi_\varepsilon - \mathbf{1}_A + \mathbf{1}_{\mathbb{R}^2\setminus A}) = \mathcal{O}(\varepsilon)\,. \tag{6.5}$$

For the first term in (6.3), using that $\int_{\mathbb{R}^2} \eta_\varepsilon^2 = 1 = \alpha_1 + \alpha_2$ and that $\int_{\mathcal{D}\cap A} \rho = \alpha_2$, we obtain

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2(\mathbf{1}_{\mathbb{R}^2} - 2\mathbf{1}_A) = \alpha_1 - \alpha_2 + \int_{A\cap\mathcal{D}} (\eta_\varepsilon^2 - \rho) + \int_{A\setminus\mathcal{D}} \eta_\varepsilon^2\,.$$

Using (2.5) we get, for $\alpha \in (1/2, 3/5)$ and $\gamma \in (1/2, 3/4)$,

$$\begin{aligned}
\int_{A\cap\mathcal{D}} (\eta_\varepsilon^2 - \rho) &= \int_{A\cap B(0,\lambda-\varepsilon^\alpha)} (\eta_\varepsilon^2 - \rho) + \int_{(A\cap\mathcal{D})\setminus B(0,\lambda-\varepsilon^\alpha)} (\eta_\varepsilon^2 - \rho) \\
&= \mathcal{O}(\varepsilon^\gamma) + \mathcal{O}(\varepsilon^\alpha)\,.
\end{aligned} \tag{6.6}$$

Moreover, from (2.7), we have $\eta_\varepsilon^2(x) \le \eta_\varepsilon^2(x_\alpha)$ in $A\setminus\mathcal{D}$, with $x_\alpha \in \partial B(0, \lambda - \varepsilon^\alpha)$. From (2.5) and (2.8) we get

$$\eta_\varepsilon^2(x) \le \eta_\varepsilon^2(x_\alpha) = \eta_\varepsilon^2(x_\alpha) - \rho(x_\alpha) + \rho(x_\alpha) = \mathcal{O}(\varepsilon^\alpha)\,,$$

so using that A is a bounded set we obtain

$$\int_{A\setminus\mathcal{D}} \eta_\varepsilon^2 = \mathcal{O}(\varepsilon^\alpha)\,. \tag{6.7}$$

For the second term in (6.3), the definitions of κ_ε and r_ε yield

$$\int_{B_0} \eta_\varepsilon^2 \kappa_\varepsilon(2 + \kappa_\varepsilon) = \frac{1}{2} r_\varepsilon + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{2\tau})\,. \tag{6.8}$$

Putting (6.5)-(6.8) in (6.3) and considering (6.2) we get

$$c_\varepsilon^{-2} \int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon = \alpha_1 - \alpha_2 + \frac{1}{2} r_\varepsilon + \mathcal{O}(\varepsilon^\beta)\,,$$

where $\beta = \min\{1, \alpha, \gamma, 2\tau\} = \min\{\alpha, \gamma\} \in (1/2, 3/5)$. Hence, (6.2) gives

$$\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon - (\alpha_1 - \alpha_2) = \left(\frac{1}{2} - (\alpha_1 - \alpha_2)\right) r_\varepsilon + \mathcal{O}(\varepsilon^\beta)\,.$$

Suppose now, without loss of generality, that $\alpha_1 - \alpha_2 \le 1/2$. The definition of r_ε and κ_ε, together with (2.4), (2.8) and $B_0 \subset\subset A \cap \mathcal{D}$, give then

$$
\begin{aligned}
|r_\varepsilon| &\geq 4\inf_{B_0} \eta_\varepsilon^2 \int_{B_0} \kappa_\varepsilon^2 + \mathcal{O}(\varepsilon) \\
&\geq 4\inf_{B_0} \eta_\varepsilon^2 \, |\ell| \, \varepsilon^\tau \int_{B_{r_0/2}(x_0)} \kappa_\varepsilon^2 + \mathcal{O}(\varepsilon) \\
&\geq c \, |\ell| \, \varepsilon^\tau + \mathcal{O}(\varepsilon) \, ,
\end{aligned}
$$

for some $c > 0$ not depending on ε. Hence, if we take $\ell = 1$ in the definition of κ_ε, for ε small enough we have

$$
\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon - (\alpha_1 - \alpha_2) \geq c' \varepsilon^\tau (1 + \varepsilon^{1-\tau} - \varepsilon^{\beta-\tau}) \, .
$$

Analogously, taking now $\ell = -1$, we get

$$
\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon - (\alpha_1 - \alpha_2) \leq c'' \varepsilon^\tau (-1 + \varepsilon^{1-\tau} + \varepsilon^{\beta-\tau}) \, .
$$

Since $\beta \in (1/2, 3/5)$, we can choose $\tau \in (1/2, \beta)$ and obtain

$$
\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon > \alpha_1 - \alpha_2 \quad \text{if} \quad \ell = 1
$$

and

$$
\int_{\mathbb{R}^2} \eta_\varepsilon^2 v_\varepsilon^2 \cos\varphi_\varepsilon < \alpha_1 - \alpha_2 \quad \text{if} \quad \ell = -1 \, .
$$

Hence, there exists $\ell_\varepsilon \in (-1, 1)$ such that for ε small enough, the associated pair $(v_\varepsilon, \varphi_\varepsilon)$ satisfy the second mass constraint in (1.8).

(Step 2 : Computing the energy). We now compute the energy of $(v_\varepsilon, \varphi_\varepsilon)$. We recall that $N_{\varepsilon\tilde{t}_\varepsilon}$ is the transition zone of φ_ε defined in (4.13). For the energy G_ε, we have that φ_ε is constant out of $N_{\varepsilon\tilde{t}_\varepsilon}$, while $v_\varepsilon = c_\varepsilon \check{v}_\varepsilon$ in $N_{\varepsilon\tilde{t}_\varepsilon}$ with $c_\varepsilon = 1 + \mathcal{O}(\varepsilon^\tau)$. Hence,

$$
\varepsilon G_\varepsilon(v_\varepsilon, \varphi_\varepsilon) = (1 + \mathcal{O}(\varepsilon^\tau)) \varepsilon G_\varepsilon(\check{v}_\varepsilon, \varphi_\varepsilon) \, . \tag{6.9}
$$

For the energy F_ε, we have that $v_\varepsilon = c_\varepsilon(1 + \kappa_\varepsilon)$ in B_0. The definition of κ_ε gives then, $|\nabla v_\varepsilon|^2 = \mathcal{O}(\varepsilon^{2\tau})$ and $\{1 - v_\varepsilon^2\}^2 = \mathcal{O}(\varepsilon^{2\tau})$. Hence,

$$
\varepsilon F_\varepsilon(v_\varepsilon \, ; \, B_0) = \mathcal{O}(\varepsilon^{2\tau-1}) = o_{\varepsilon\to 0}(1) \, . \tag{6.10}
$$

In $\mathbb{R}^2 \backslash (N_\varepsilon \cap B_0)$, we have that $v_\varepsilon = c_\varepsilon$, so $|\nabla v_\varepsilon| = 0$ and $\{1 - v_\varepsilon^2\}^2 = \mathcal{O}(\varepsilon^{2\tau})$. As before we get

$$
\varepsilon F_\varepsilon(v_\varepsilon \, ; \, \mathbb{R}^2 \backslash (N_\varepsilon \cap B_0)) = \mathcal{O}(\varepsilon^{2\tau-1}) = o_{\varepsilon\to 0}(1) \, . \tag{6.11}
$$

In N_ε, we have that $v_\varepsilon = c_\varepsilon \check{v}_\varepsilon$. Hence, $|\nabla v_\varepsilon|^2 = (1 + \mathcal{O}(\varepsilon^\tau)) |\nabla \check{v}_\varepsilon|^2$ and $\{1 - v_\varepsilon^2\}^2 = (1 + \mathcal{O}(\varepsilon^\tau)) \{1 - \check{v}_\varepsilon^2\}^2 + \mathcal{O}(\varepsilon^\tau)$, which gives

$$
\begin{aligned}
\varepsilon F_\varepsilon(v_\varepsilon \, ; \, N_\varepsilon) &= (1 + \mathcal{O}(\varepsilon^\tau)) \, \varepsilon F_\varepsilon(\check{v}_\varepsilon \, ; \, N_\varepsilon) + \mathcal{O}(\varepsilon^\tau) \varepsilon^{-1} \mathcal{L}^2(N_\varepsilon) \\
&= (1 + \mathcal{O}(\varepsilon^\tau)) \, \varepsilon F_\varepsilon(\check{v}_\varepsilon \, ; \, N_\varepsilon) + o_{\varepsilon\to 0}(1) \, .
\end{aligned} \tag{6.12}
$$

Since $\check{v}_\varepsilon$ is constant out of N_ε, we have $F_\varepsilon(\check{v}_\varepsilon) = F_\varepsilon(\check{v}_\varepsilon\,;\,N_\varepsilon)$. Putting together (6.1) and (6.9)-(6.12), we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) = \limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(\check{v}_\varepsilon, \varphi_\varepsilon) \leq \mathcal{F}(\varphi)\,. \tag{6.13}$$

(Step 3 : identification of $(v_\varepsilon, \varphi_\varepsilon)$*)* The pairs of test functions satisfies the hypothesis from Proposition 3.1**(ii)**, so defining $(u_{1,\varepsilon}, u_{2,\varepsilon})$ by (3.4) we have $(u_{1,\varepsilon}, u_{2,\varepsilon}) \in \mathcal{H}$ and $u_{1,\varepsilon}^2 + u_{2,\varepsilon}^2 > 0$. Hence, after Proposition (3.2) relation (1.9) holds, and (6.13) yield

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) = \limsup_{\varepsilon\to 0} \varepsilon \mathcal{F}_\varepsilon(v_\varepsilon, \varphi_\varepsilon) \leq \mathcal{F}(\varphi)\,.$$

□

6.3 Proof of Corollary 1.2:

Let $\tilde{\varphi} \in X$ with $\mathcal{F}(\tilde{\varphi}) < +\infty$. From the upper bound inequality in Theorem 1.1, there is a sequence $(\tilde{u}_{1,\varepsilon}, \tilde{u}_{2,\varepsilon}) \in \mathcal{H}$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(\tilde{u}_{1,\varepsilon}, \tilde{u}_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) \leq \mathcal{F}(\tilde{\varphi})\,.$$

Since $(u_{1,\varepsilon}, u_{2,\varepsilon})$ minimize $\mathcal{E}_\varepsilon$ in $\mathcal{H}$, the previous inequality yields

$$\limsup_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) \leq \mathcal{F}(\tilde{\varphi})\,, \tag{6.14}$$

so in particular $(u_{1,\varepsilon}, u_{2,\varepsilon})$ satisfy (1.16). Hence, from the compactness and the lower bound inequality in Theorem 1.1, there is $\varphi \in X$ and a subsequence $(u_{1,\varepsilon'}, u_{2,\varepsilon'})$ with

$$\liminf_{\varepsilon'\to 0} \varepsilon' \left(\mathcal{E}_{\varepsilon'}(u_{1,\varepsilon'}, u_{2,\varepsilon'}) - E_{\varepsilon'}(\eta_{\varepsilon'})\right) \geq \mathcal{F}(\varphi)\,.$$

This inequality is verified for every subsequence of $(u_{1,\varepsilon}, u_{2,\varepsilon})$, so we have

$$\liminf_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) \geq \mathcal{F}(\varphi)\,. \tag{6.15}$$

From (6.14) and (6.15), we obtain

$$\mathcal{F}(\tilde{\varphi}) \geq \limsup_{\varepsilon\to 0} \varepsilon \mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) \geq \liminf_{\varepsilon\to 0} \varepsilon \mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) \geq \mathcal{F}(\varphi)\,, \tag{6.16}$$

so $\mathcal{F}(\varphi) = \inf_X \mathcal{F}$. Taking $\tilde{\varphi} = \varphi$ in (6.16) yields

$$\lim_{\varepsilon\to 0} \varepsilon \left(\mathcal{E}_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) - E_\varepsilon(\eta_\varepsilon)\right) = \inf_X \mathcal{F}\,.$$

□

6.4 Proof of Corollary 1.3

We start proving that when α_1 is not to close to 0 or 1, the minimizers of $\mathcal{F}$ in X are not radially symmetric. We show that for any radially symmetric $\varphi \in X$, $\mathcal{F}(\varphi) > \mathcal{F}(\varphi_{ds})$, where the support of $\varphi_{ds} \in X$ is a disk sector. We first prove this for functions such that $\{\varphi = 0\}$ is a disk or an annulus. Then, we generalize by induction the result to radial functions such that $\{\varphi = 0\}$ is composed of a finite number of connected components. We conclude then by approximating any radially symmetric $\varphi \in X$ by this kind of functions.

We recall that ρ is given in (1.3) and that X is the space of functions $\varphi \in BV_{loc}(\mathcal{D}; \{0, \pi\})$ such that

$$\int_{\{\varphi=0\}} \rho = \alpha_1 \,. \tag{6.17}$$

If $\varphi_{ds} \in X$ is such that $\{\varphi_{ds} = 0\}$ is a disk sector, we easily compute

$$\frac{\mathcal{F}(\varphi_{ds})}{8\sigma} = \frac{3}{16} \,.$$

For $0 \leq R^- \leq R^+ \leq \lambda$ we denote $A(R^-, R^+)$ the annulus of center the origin, inner radius R^- and outer radius R^+.

If $\varphi_\alpha \in X$ is such that $\{\varphi = 0\} = A(0, R_\alpha)$ and $\int_{A(0,R_\alpha)} \rho = \alpha$, then $R_\alpha = \lambda(1 - \sqrt{\alpha})^{1/2}$ and

$$\frac{\mathcal{F}(\varphi_\alpha)}{8\sigma} = f(\alpha) \,, \tag{6.18}$$

where $f : [0, 1] \to \mathbb{R}_+$ is the concave function $f(\alpha) = (1 - \alpha)^{3/4}(1 - \sqrt{1 - \alpha})^{1/2}$. We see that there exists

$$\delta_0 \approx 0.1486$$

such that if $\alpha \in [\delta_0, 1 - \delta_0]$, then $f(\alpha) > 3/16$.

Proposition 6.1. *If $\alpha_1 \in [\delta_0, 1 - \delta_0]$, then the minimizers of $\mathcal{F}$ in X are not radially symmetric.*

Proof: *(Step 1)* Let $R \in (0, \lambda)$ and consider $\varphi_1^d \in X$ such that $\{\varphi_1^d = 0\} = A(0, R)$. From (6.17), we have that $\mathcal{F}(\varphi_1^d)/8\sigma = f(\alpha_1)$ so (6.18) yields

$$\mathcal{F}(\varphi_1^d) > \mathcal{F}(\varphi_{ds}) \,. \tag{6.19}$$

Since $\alpha_2 = 1 - \alpha_1 \in [\delta_0, 1 - \delta_0]$, the similar inequality holds if $\{\varphi_1^d = 0\} = A(R, \lambda)$.

Consider now $\varphi_1^a \in X$ such that $\{\varphi_1^a = 0\} = A(R_1, R_2)$, with $0 < R_1 < R_2 < \lambda$. Writing

$$\beta_1 = \int_{A(0,R_1)} \rho \,, \qquad \beta_2 = \int_{A(R_1,R_2)} \rho \quad \text{and} \quad \beta_3 = \int_{A(R_2,\lambda)} \rho \,,$$

we compute

$$\frac{\mathcal{F}(\varphi_1^a)}{8\sigma} = f(\beta_1) + f(\beta_1 + \beta_2)\,.$$

From (6.17), we have that $\beta_2 = \alpha_1$ and $\beta_1 + \beta_3 = \alpha_2$ so

$$\frac{\mathcal{F}(\varphi_1^a)}{8\sigma} = f(\beta_1) + f(\beta_1 + \alpha_1)\,.$$

The right hand size of the previous equality is a concave function of β_1 and the value of β_1 may vary between 0 and α_2. If $\beta_1 = 0$ then $\mathcal{F}(\varphi_1^a)/8\sigma = f(\alpha_1)$. If $\beta_1 = \alpha_2$, since $\alpha_1 + \alpha_2 = 1$ we find $\mathcal{F}(\varphi_1^a)/8\sigma = f(\alpha_2)$. We derive

$$\mathcal{F}(\varphi_1^a) > \mathcal{F}(\varphi_{ds})\,. \tag{6.20}$$

(Step 2) Let $n \in \mathbb{N}^*$ and consider $\varphi_n \in X$ such that

$$\{\varphi_n = 0\} = \bigcup_{j=1}^{n} A_{2j}\,,$$

with $A_{2j} = A(R_{2j}^-, R_{2j}^+)$ and

$$0 \le R_{2j-2}^- < R_{2j-2}^+ < R_{2j}^- < R_{2j}^+ \le \lambda$$

for $2 \le j \le n$. We write $\beta_{2j} = \int_{A_{2j}} \rho$, $\beta_1 = \int_{A(0,R_2^-)} \rho$, $\beta_{2n+1} = \int_{A(R_{2n}^+,\lambda)} \rho$ and

$$\beta_{2j+1} = \int_{A(R_{2j}^+, R_{2j+2}^-)} \rho$$

for $1 \le j \le n-1$. Notice that we allow $A(0, R_2^-)$ or $A(R_{2n}^+, \lambda)$ to be empty, but this only implies that $\beta_1 = 0$ or $\beta_{2n+1} = 0$. With this notation, we have

$$\sum_{i=1}^{n} \beta_{2i} = \alpha_1\,, \qquad \sum_{i=1}^{n} \beta_{2i+1} = \alpha_2 \tag{6.21}$$

and

$$\frac{\mathcal{F}(\varphi_n)}{8\sigma} = \sum_{j=1}^{2n} f\left(\sum_{i=1}^{j} \beta_i\right) =: g_n(\beta_1, \cdots, \beta_{2n})\,.$$

By induction, we are going to prove the following property:

$$(\mathcal{P}_n) \qquad \forall\, \beta_1, \cdots, \beta_{2n+1} \in [0,1] \text{ such that } \sum_{i=1}^{n} \beta_{2i} = \alpha_1 \text{ and } \sum_{i=1}^{n} \beta_{2i+1} = \alpha_2\,,$$
$$g_n(\beta_1, \cdots, \beta_{2n}) > \frac{\mathcal{F}(\varphi_{ds})}{8\sigma}\,.$$

If $n = 1$ we are in one of the three cases analyzed in Step 1, so (6.19) and (6.20) yield $(\mathcal{P}_1)$.

Let us assume that $(\mathcal{P}_n)$ holds and consider $\beta_1, \cdots, \beta_{2n+3} \in [0,1]$ such that

$$\sum_{i=1}^{n+1} \beta_{2i} = \alpha_1 \quad \text{and} \quad \sum_{i=1}^{n+1} \beta_{2i+1} = \alpha_2 \,. \tag{6.22}$$

We have

$$g_{n+1}(\beta_1, \cdots, \beta_{2n+2}) = \sum_{j=1}^{2n} f\left(\sum_{i=1}^{j} \beta_i\right) + f\left(\sum_{i=1}^{2n+1} \beta_i\right) + f\left(\sum_{i=1}^{2n+2} \beta_i\right) .$$

The right hand side of the previous equality is a concave function of β_{2n+2}. The value of β_{2n+2} may vary between 0 and α_1. Suppose first that $\beta_{2n+2} = 0$. Then, defining

$$\tilde{\beta}_j = \beta_j \quad \text{if} \quad j = 1, \cdots, 2n \qquad \text{and} \qquad \tilde{\beta}_{2n+1} = \beta_{2n+1} + \beta_{2n+3} \,,$$

the $\tilde{\beta}_i$'s satisfy (6.21) and we have

$$g_{n+1}(\beta_1, \cdots, \beta_{2n+2}) \geq \sum_{j=1}^{2n} f\left(\sum_{i=1}^{j} \beta_i\right) = g_n(\tilde{\beta}_1, \cdots, \tilde{\beta}_{2n}) \,.$$

Hence, $(\mathcal{P}_n)$ yields $g_{n+1}(\beta_1, \cdots, \beta_{2n+2}) > \mathcal{F}(\varphi_{ds})/8\sigma$.

Suppose now that $\beta_{2n+2} = \alpha_1$. From (6.22) this implies $\beta_{2j} = 0$ for every $j = 1, \cdots, n$. Then, defining

$$\tilde{\beta}_1 = \sum_{j=1}^{2n+1} \beta_j \,, \qquad \tilde{\beta}_2 = \beta_{2n+2} \qquad \text{and} \qquad \tilde{\beta}_3 = \beta_{2n+3} \,,$$

the $\tilde{\beta}_i$'s satisfy (6.21) and we have

$$\begin{aligned} g_{n+1}(\beta_1, \cdots, \beta_{2n+2}) &\geq f\left(\sum_{i=1}^{2n+1} \beta_i\right) + f\left(\sum_{i=1}^{2n+2} \beta_i\right) \\ &= f(\tilde{\beta}_1) + f(\tilde{\beta}_1 + \tilde{\beta}_2) \\ &= g_1(\tilde{\beta}_1, \tilde{\beta}_2) \,. \end{aligned}$$

Hence, $(\mathcal{P}_1)$ yields $g_{n+1}(\beta_1, \cdots, \beta_{2n+2}) > \mathcal{F}(\varphi_{ds})/8\sigma$. We derive that the result holds for all the possible values of β_{2n+2}.

We have proved that if $\varphi_n \in X$ is radial and its support has a finite number of connected components, then

$$\mathcal{F}(\varphi_n) > \mathcal{F}(\varphi_{ds}) \,. \tag{6.23}$$

(Step 3) Suppose now that $\varphi \in X$ is a radially symmetric function such that $\{\varphi = 0\}$ has an infinite number of connected components. Since φ has locally finite perimeter in $\mathcal{D}$, $\{\varphi = 0\}$ is the union of a countable family of disjoints annuli. We write

$$\{\varphi = 0\} = \bigcup_{j \in \mathbb{Z}} A_{2j}$$

with $A_{2j} = A(R_{2j}^-, R_{2j}^+)$ such that

$$0 < R_{2j}^- < R_{2j}^+ < R_{2j+2}^- < R_{2j+2}^+ < \lambda \,. \tag{6.24}$$

For every $n \in \mathbb{N}$, we define a function $\varphi_n : \mathcal{D} \to \{0, \pi\}$ by

$$\{\varphi_n = 0\} = \bigcup_{j=-n}^{n} A_{2j} \bigcup \tilde{A}_{2n+2} \bigcup \tilde{A}_{-2n-2} \,,$$

such that

$$\tilde{A}_{2n+2} = A(L_n^+, \lambda) \quad \text{and} \quad \tilde{A}_{-2n-2} = A(0, L_n^-)$$

with L_n^-, L_n^+ to be chosen next. If $(L_n^-, L_n^+) = (0, \lambda)$, then (6.24) gives

$$\int_{\{\varphi_n=0\}} \rho = \sum_{j=-n}^{n} \int_{A_{2j}} \rho < \int_{\{\varphi=0\}} \rho \,.$$

Similarly if $(L_n^-, L_n^+) = (R_{-2n}^-, R_{2n}^+)$, then

$$\int_{\{\varphi_n=0\}} \rho = \sum_{j\in\mathbb{Z}} \int_{A_{2j}} \rho + \sum_{j\geq n} \int_{A(R_{2j}^+, R_{2j+2}^-)} \rho + \sum_{j\leq -n} \int_{A(R_{2j-2}^+, R_{2j}^-)} \rho > \int_{\{\varphi=0\}} \rho \,.$$

Hence, by continuity there is a pair $(L_n^-, L_n^+) \in (0, R_{-2n}^-) \times (R_{2n}^+, \lambda)$ such that $\int_{\{\varphi_n=0\}} \rho = \int_{\{\varphi=0\}} \rho = \alpha_1$. Clearly $\varphi_n \in BV_{loc}(\mathcal{D})$, so $\varphi_n \in X$. Moreover, (6.24) yields

$$\lim_{n\to\infty} L_n^- = 0 \quad \text{and} \quad \lim_{n\to\infty} L_n^+ = \lambda \,. \tag{6.25}$$

We have

$$\mathcal{F}(\varphi) = \sum_{j\in\mathbb{Z}} \int_{\partial B(0,R_{2j}^+)} \rho^{3/2} \, d\mathcal{H}^1 \,,$$

and since ρ is radially symmetric

$$\mathcal{F}(\varphi_n) \;=\; \sum_{j=-n}^{n} \int_{\partial B(0,R_{2j}^+)} \rho^{3/2} \, d\mathcal{H}^1 + 2\pi \left(\rho^{3/2}(L_n^+) \, L_n^+ + \rho^{3/2}(L_n^-) \, L_n^- \right) \,.$$

From (6.25), the last term in the previous equality goes to zero as $n \to +\infty$, so $\lim_{n\to\infty} \mathcal{F}(\varphi_n) = \mathcal{F}(\varphi)$. Hence, since $\{\varphi_n = 0\}$ has a finite number of connected components, (6.23) yields $\mathcal{F}(\varphi) > \mathcal{F}(\varphi_{ds})$, which ends the proof.

□

Proof of Corollary 1.3: Suppose that $\alpha_1 \in [\delta_0, 1-\delta_0]$ and that $\{(u_{1,\varepsilon}, u_{2,\varepsilon})\}_{\varepsilon>0}$ is a sequence of radially symmetric pairs such that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ minimizes $\mathcal{E}_\varepsilon$ under the mass constraints (1.5). Then, φ_ε defined by (3.7) is also radially symmetric. Consider $\varphi_{\varepsilon,0}$, the restriction of φ_ε to a slice of $\mathcal{D}$ passing through 0. From Proposition 5.1, $\varphi_{\varepsilon,0}$ belongs to $SBV_{loc}([0,\lambda]\,;\,\{0,\pi\})$ and converges in $L^1_{loc}([0,\lambda])$ to φ_0. Hence, φ_ε converges in $L^1_{loc}(\mathcal{D})$ to the radial function φ given by $\varphi(x) = \varphi_0(|x|)$. From Corollary 1.2, we know that φ

minimizes $\mathcal{F}$ over X, which yields a contradiction with Proposition 6.1.

$\square$

7 Appendix

We end this article given the proof of Lemma 4.2, which is essentially the same of Lemma 4.3 in [10], which in turn is a generalization of Lemma 1 in [26]. For completeness we give here the details of the proof.

Proof of Lemma 4.2: *(Step 1)* Suppose first that $\mathcal{D}\cap A$ and $\mathcal{D}\backslash A$ have both non empty interior and let

$$\overline{B(x_1,\delta)} \subset \mathcal{D}\cap A \qquad \text{and} \qquad \overline{B(x_2,\delta)} \subset \mathcal{D}\backslash A\,. \tag{7.1}$$

We first approximate A by sets of finite perimeter in $\mathcal{D}$. For $k \geq 2$ we define $\mathcal{D}_k = \mathcal{D}\cap B(0,\lambda(1-{}^1\!/\!_k))$ and $A'_k = A\cap\mathcal{D}_k$. We have that $\partial_* A'_k \subset (\partial_* A\cap\mathcal{D}_k)\cup\partial\mathcal{D}_k$, so

$$\int_{\partial_* A'_k} \rho^{3/2}\, d\mathcal{H}^1 \leq \int_{\partial_* A\cap\mathcal{D}_k} \rho^{3/2}\, d\mathcal{H}^1 + \int_{\partial\mathcal{D}_k} \rho^{3/2}\, d\mathcal{H}^1\,.$$

Using Lebesgue dominated convergence theorem, the first term in the right hand side of the inequality converges to $\int_{\partial_* A} \rho^{3/2}\, d\mathcal{H}^1 < +\infty$. The definition of $\mathcal{D}_k$ and (2.8) yield

$$\int_{\partial\mathcal{D}_k} \rho^{3/2}\, d\mathcal{H}^1 \leq \|\rho\|^{3/2}_{L^\infty(\partial\mathcal{D}_k)}\, \mathcal{H}^1(\partial\mathcal{D}_k) \leq \Big(\frac{2\lambda^2}{k}\Big)^{3/2} \mathcal{H}^1(\partial\mathcal{D}) = o_{k\to\infty}(1)\,.$$

Hence,

$$\lim_{k\to\infty} \int_{\partial_* A'_k} \rho^{3/2}\, d\mathcal{H}^1 \leq \int_{\partial_* A} \rho^{3/2}\, d\mathcal{H}^1\,. \tag{7.2}$$

(Step 2) Since A'_k has finite perimeter in $\mathcal{D}$, it can be approximated (see the proof of Lemma 1 in [26]) by open bounded sets $\tilde{A}_k$, such that

$$\mathcal{L}^2(\tilde{A}_k \Delta A'_k) \;\leq\; \frac{1}{k} \tag{7.3}$$

$$A'_k \subset \tilde{A}_k + B(0,{}^1\!/\!_k) \qquad \text{and} \qquad \tilde{A}_k \subset A'_k + B(0,{}^1\!/\!_k) \tag{7.4}$$

$$\mathcal{H}^1(\partial\tilde{A}_k\cap\partial\mathcal{D}) \;=\; 0\,. \tag{7.5}$$

The definition of A'_k and (7.3) imply **(i)**. Using (7.1) and (7.4), for large enough k we get

$$B(x_1,\delta) \subset \tilde{A}_k \qquad \text{and} \qquad B(x_2,\delta)\subset \mathcal{D}\backslash\tilde{A}_k\,. \tag{7.6}$$

Moreover, using (ii) from Proposition 2.3 in [10] and the fact that $\tilde{A}_k$ belongs to a sequence $\tilde{A}^n_k$ such that $\|\tilde{A}^n_k\|_{BV(\mathcal{D})} \to \|A'_k\|_{BV(\mathcal{D})}$ as $n\to 0$, we have

$$\int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{\tilde{A}_k}| \leq \int_{\mathcal{D}} \rho^{3/2}|D\mathbf{1}_{A'_k}| + \frac{1}{k}\,. \tag{7.7}$$

Also, the definition of A'_k and (7.3) yield

$$\gamma_k := \int_{\tilde{A}_k} \rho - \int_A \rho = o_{k\to\infty}(1)\,. \tag{7.8}$$

(Step 3) Now, we set

$$A_k = \begin{cases} \tilde{A}_k \backslash B(x_1, r_{1,k}) & \text{if} \quad \gamma_k > 0 \\ \tilde{A}_k & \text{if} \quad \gamma_k = 0 \\ \tilde{A}_k \cup B(x_1, r_{2,k}) & \text{if} \quad \gamma_k < 0 \end{cases}\,,$$

where $r_{1,k}$ and $r_{2,k}$ are chosen to satisfy

$$\int_{B(x_1,r_{1,k})} \rho = \int_{B(x_2,r_{2,k})} \rho = \gamma_k\,.$$

Since $r \mapsto \int_{B(x_{1,2},r)} \rho$ is continuous and decreasing for $r \in (0,\delta)$, $r_{1,k}$ and $r_{2,k}$ are unique and tend to zero as $k \to \infty$. Then, we derive from (7.6) and (7.8), for large enough k, that

$$\int_{\mathcal{D}\cap A_k} \rho = \int_{\tilde{A}_k} \rho - \gamma_k = \int_A \rho\,.$$

Moreover, from (7.5) and (7.6), we have $\mathcal{H}^1(\partial A_k) = 0$ for k large enough, so **(ii)** is proved. Using again (7.6) we obtain

$$\int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{A_k}| \le \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{\tilde{A}_k}| + \|\rho\|_\infty\, H^1(\partial B(x_1, r_{1,k}) \cup \partial B(x_2, r_{1,2}))\,. \tag{7.9}$$

Hence, using (7.7), we obtain

$$\int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{A_k}| \le \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{A'_k}| + o_{k\to\infty}(1)\,,$$

so (7.2) gives

$$\limsup_{k\to\infty} \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{A_k}| \le \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_A|$$

We have proved **(iii)**.

(Step 4) We now remove the condition that $\mathcal{D}\cap A$ and $\mathcal{D}\backslash A$ have no empty interior. First, we notice that $\mathcal{L}^2(\mathcal{D}\cap A) = 0$ and $\mathcal{L}^2(\mathcal{D}\backslash A) = 0$ are not possible because of the mass constraints in (1.8). Hence, there exists x_1 a point of density of $\mathcal{D}\cap A$ and x_2 a point of density of $\mathcal{D}\backslash A$. Consider the function

$$\Phi(\delta_1, \delta_2) = \int_{A_{12}} \rho - \int_A \rho\,,$$

where $A_{12} = A \cup B(x_1, \delta_1) \backslash B(x_2, \delta_2)$. Since $\rho > 0$ in $\mathcal{D}$, for any $\delta > 0$ we have

$$\Phi(\delta, 0) > 0 \qquad \text{and} \qquad \Phi(0, \delta) < 0\,.$$

Since Φ is continuous, there is $t = t_\delta \in (0,1)$ such that $\Phi(t\delta, (1-t)\delta) = 0$. Define $A_\delta = A \cup B(x_1, (1-t)\delta) \backslash B(x_2, t\delta)$ and $\varphi = \pi \mathbf{1}_{A_\delta}$. Both $\mathcal{D}\cap A_\delta$ and $\mathcal{D}\backslash A_\delta$ have no empty

interior and $\int_{A_\delta} \rho = \int_A \rho$. Moreover $\mathcal{L}^2(A_\delta \Delta A) \to 0$ as $\delta \to 0$, and using an inequality similar to (7.9), we get

$$\limsup_{\delta\to 0} \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_{A_\delta}| \le \int_{\mathcal{D}} \rho^{3/2} |D\mathbf{1}_A| \,.$$

Finally, for each A_δ we apply the construction from steps 1-3 and conclude thanks a diagonal argument, see Corollary 1.16 in [7].

□

Acknowledgements The second author would like to acknowledge discussions with Guy Bouchitté, Pierre Seppecher and Duvan Henao. We would like to thank Clément Gallo.

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